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INTRODUCTORY MATHEMATICAL ANALYSIS

BY

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FIRST EDITION

FIRST THOUSAND

NEW YORK

JOHN WILEY & SONS, Inc.

London: CHAPMAN & HALL, Limited

1919

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A.

PREFACE

The present course is the result of several years of study and trial in the classroom in an effort to make an introduction to college mathematics more effective, rational and better suited to its place in a scheme of education under modern conditions of life. A broader field has been attempted than is customary in books of its class. This is made possible by certain principles which controlled the construction of the text.

One principle on which the course is built is correlation by topics. For example, all methods of calculation have been associated in one chapter and early in the course in order to be available for use in the sequel.

The function idea has also been emphasized and used as a means of correlation.

Brevity and directness of treatment have contributed to reduce the size of the book.

An effort has been made to keep in view of the student the steps in the development of the subject and to point out useful contacts of mathematics with affairs.

The first two chapters are intended to be used for review and reference at the discretion of the instructor.

Graphic representation and its uses have been given considerable attention. The simple cases of determining empirical formulæ give a very valuable drill in the solution of simultaneous equations and a foundation for later work in the laboratory.

The treatment of the trigonometric functions is brief, direct and in some respects more advanced in style than is customary in current texts in trigonometry which are constructed mostly from the secondary school standpoint. Large use of the func-

tions is made in a variety of applications in immediately following chapters.

More than usual attention is given to vectors. The value and convenience of vector methods in science and engineering seem to justify this emphasis. The part dealing with vector products and the problems depending on it may, however, be omitted without inconvenience in later chapters.

The chapter on series may seem a little heavy for freshmen but it comes in the second half of the course and is directly applied to functions within the experience of the student in the preceding text.

What is given on differential and integral calculus is intended as an introduction for those who are to take the regular and fuller course in calculus. For those who are not to continue their mathematics it will furnish an introduction to the methods of calculus and some important definite applications. The integral has first been regarded as the inverse of the derivative and nothing is said about the differential. This seems natural and in accord with the idea of the solution of differential equations under many actual conditions where a function is sought whose derivative is given. Following, the integral is regarded as a summation of elements and some further applications are introduced. In the list of integrals for reference both the inverse and the differential forms are given.

In general no effort at rigor beyond reasonable conviction has been attempted. Proofs have been given for some theorems that many teachers may prefer to regard as assumptions. These proofs may, therefore, be omitted at the discretion of the teacher. A number of what appear as theorems in some texts are here given as exercises. For this reason it is recommended that each student be held for practically all the exercises appearing regularly through the text. Selections may be made at the instructor's discretion from the exercises at the end of each chapter.

The teacher will find an opportunity for originality in developing the text and at times a necessity for more details.

The entire course has been given several times to classes meeting daily during a period equivalent to four terms of three months each. The work can be covered in less time.

The authors take this opportunity to acknowledge their obligation to Dean H. B. Meller, of the University of Pittsburgh, for affording the opportunity to have the course tried out in the classroom and for a number of problems.

If this book shall contribute toward making more satisfactory and more economical the modicum of college mathematics the authors will feel well repaid for the considerable labor of its composition.

W. P. W.
L. C. P.

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INTRODUCTORY MATHEMATICAL ANALYSIS

CHAPTER I

REVIEW OF ALGEBRA

1. Signs of aggregation are used to indicate that an operation is to extend to each term of a group of terms or to separate or to specify terms or factors in an algebraic expression. Thus:

1. $3\,a\,(b + c + d)$ shows that each term inside the "( )" is to be multiplied by the factor, $3\,a$, which precedes the "( )", or $3\,a\,(b + c + d)$ is equivalent to $3\,ab + 3\,ac + 3\,ad$.

2. $\sqrt{6\,a + 3\,b - 24}$ shows by the "——" (vinculum) that the "$\sqrt{}$" (square root) is to be taken of the expression under the "——" considered as a whole, and not the square root of each term separately.

3. $a - (2\,c - 4\,b + d)$ or $a - \overline{2\,c - 4\,b + d}$ shows that the terms in "( )" or under "——" are all to be subtracted from a. Since we may either subtract these singly or first combine them and subtract the combined result, we may express the operation as,

$$\begin{aligned} a - (2\,c - 4\,b + d) &= a - \overline{2\,c - 4\,b + d} \\ &= a - 2\,c - (-4\,b) - d = a - 2\,c + 4\,b - d. \end{aligned}$$

4. $(3 + 6\,a - 4\,b)\,(a - 5\,c + d)$ shows that the combined value of the terms in each "( )" is to be multiplied by the other. This is accomplished by multiplying every term in one "( )" by each term in the other "( )" and reducing the result to the simplest form.

5. $a + b - c + d + \sqrt{c} - e$ may, by using a sign of aggregation, say a "( )," be written as $a + b + d - (c + e - \sqrt{c})$.

Remark. — When a sign of aggregation is preceded by a $\begin{matrix}\text{plus}\\ \text{minus}\end{matrix}$ sign, the sign of aggregation may be removed $\begin{matrix}\text{without changing}\\ \text{by changing}\end{matrix}$ the signs of the terms within the sign of aggregation. We must not confuse signs of aggregation with signs of operation. Thus, in the expression $-\sqrt{a-b}$, the "$\sqrt{\ }$" is not a sign of aggregation, but shows that the square root is to be taken of the entire expression under the "——" considered as a whole. The minus sign before the "$\sqrt{\ }$" is not to be regarded as preceding the sign of aggregation and therefore the "——" in the expression cannot be removed by the usual rule. The operation of square root must first be performed. Similarly, other cases are to be handled.

1. $12x - (9x + 7x) = ?$; $ab + \overline{3 - ab} = ?$
2. $xy - \sqrt{x^2 - 2x + 1} = ?$; $x^2y^2 - \overline{x^2y^2 - 1} = ?$
3. $a - (a + b - (c - d + e - a) + c) - b - c + d - e = ?$
4. $(ab - (cd + 1))\,(ab - (cd - 1)) = ?$
5. $(2x^2 + (3x - 1)\,(4x + 5))\,(5x^2 - (4x + 3)\,(x - 2)) = ?$
6. $x - \{-12y - [2x + (-4y - (-7x - 5y) - \overline{6x - 9y - 8x + y})]\} = ?$
7. $(a^2 - (b^2 + c^2))\,(a^2 - (b^2 - c^2)) = ?$
8. $20a\,(10a - (6a - c - (5a - b)) + c) = ?$
9. $(\sqrt{9 - 6x^2 + x^4} - \sqrt{25 - 10a + a^2})\,(3 - \sqrt{25}) = ?$

2. The **index laws** may be stated in the form of equations as follows:

1. $a^m \cdot a^n = a^{m+n}$.
2. $a^m \div a^n = a^{m-n}$.
3. $a^m \div a^m = 1$, also $a^m \div a^m = a^{m-m} = a^0$.
   $\therefore a^0 = 1$ for any value of a.
4. $1 \div a^m = a^0 \div a^m = a^{0-m} = a^{-m}$, but $1 \div a^m = 1/a^m$.
   $\therefore a^{-m} = 1/a^m$.
5. $(a^m)^n = a^{mn}$.
6. $(ab)^n = a^n b^n$.

Perform the indicated operations with the expressions just as they are written and change the result so that all exponents shall be positive.

1. $(a^2x^{-1} + 3\,a^3x^{-2})\,(4\,a^{-1} - 5\,x^{-1} + 6\,ax^{-2}) = ?$
2. $(a^0 - a^{-1})\,(a^0 - a^{-2})\,(a^0 - a^{-3}) = ?$
3. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})\,(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) = ?$
4. $(18\,y^{-2} + 23 + x^{-\frac{1}{2}}y + 6\,x^{-1}y^2) \div (3\,x^{\frac{3}{4}}y^{-1} + x^{\frac{1}{4}} - 2\,x^{-\frac{1}{4}}y) = ?$
5. $(x - 2\,y)^5 = ?$
6. $(3\,a^{\frac{3}{2}}b^{\frac{1}{3}} + 4\,ab^{\frac{2}{3}} - a^{\frac{1}{2}}b)\,(6\,a^{\frac{1}{4}}b^{-\frac{2}{3}} - 8\,a^{-\frac{1}{4}}b^{-\frac{1}{3}} - 2\,a^{-\frac{3}{4}}) = ?$
7. $(8\,x^2 - 2\,x - 3)/(12\,x^2 - 25\,x - 12) = ?$

When $x = \frac{3}{4}$, (a) by direct substitution, (b) by carrying out the division and substituting in the quotient.

8. $(x^p - 3\,x^{p-1} + 4\,x^{p-2} - 6\,x^{p-3} + 5\,x^{p-4})\,(2\,x^3 - x^2 + x) = ?$
9. $(x^5 - y^5) \div (x - y) = ?$
10. $(6\,x^{m-n+2} - x^{m-n+1} - 22\,x^{m-n} - 19\,x^{m-n-1} - 4\,x^{m-n-2})$ $\div\,(3\,x^{3-n} + 4\,x^{2-n} + x^{-n}) = ?$

3. Some important cases of **factoring** depend on the following:

1. $(a + b)\,(a + b) = (a + b)^2 = a^2 + 2\,ab + b^2.$
2. $(a - b)\,(a - b) = (a - b)^2 = a^2 - 2\,ab + b^2.$
3. $(a + b)\,(a - b) = a^2 - b^2.$
4. $(a + x)\,(a + y) = a^2 + (x + y)\,a + xy.$
5. $(a + b)\,(a^2 - ab + b^2) = a^3 + b^3.$
6. $(a - b)\,(a^2 + ab + b^2) = a^3 - b^3.$
7. $(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3.$
8. $(a + b)^n = a^n + na^{n-1}b + (n\,(n - 1)/1 \cdot 2)\,a^{n-2}b^2 + (n\,(n-1)\,(n-2)/1 \cdot 2 \cdot 3)\,a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$
9. $(a^n - b^n) = (a - b)\,(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$
10. $ax - ay + bx - by = a\,(x - y) + b\,(x - y) = (a + b)\,(x - y).$
11. $(a \pm b \pm c)^2 = a^2 + b^2 + c^2 \pm 2\,ab \pm 2\,ac \pm 2\,bc.$

Factor the following:

1. $20\,ab - 28\,ad - 5\,bc + 7\,cd.$
2. $49\,x^6 - 168\,x^3y + 144\,y^2.$

3. $6xy + 16z^2 - 9y^2 - x^2$.
4. $a^2 - 2ab + b^2 - c^2$.
5. $a^2 - 10a - 75$.
6. $a^2 - 2ab + b^2 - 3a + 3b + 2$.
7. $x^2 + 2xy - 99y^2$.
8. $36x^2 + 12x - 35$.
9. $x^3 - 3x^2 + 3x - 1$.
10. $a^3 - 6a^2 + 18a - 27$.
11. $8x^3 - 27y^3$.
12. $x^3 - 1$.
13. $a^9 - 1$.
14. $a^2 - m^2 + a + m$.
15. $(8x^3 - 27) - (2x - 3)(4x^2 + 4x - 6)$.
16. $x^3 + 2x^2y + 2xy^2 + y^3$.
17. $(a^2 + 6a + 8)^2 - 14(a^2 + 6a + 8) - 15$.
18. $x^8 - x^5 - 32x^3 + 32$.
19. $18x^5b + 24x^3b^3 + 8xb^5$.
20. $27x^3 - 108x^2y + 144xy^2 - 64y^3$.
21. $21a^2 + 23ab + 6b^2$.
22. $10a^2 - 39a + 14$.
23. $95 - 14x^2 - x^4$.
24. $x^9 + 512$.
25. $5x + 25xy - 70x^2y^2$.

4. The **lowest common multiple** of two or more expressions is that expression of lowest degree that is exactly divisible by each of the given expressions. Thus:

1. a^2b^3c is the lowest common multiple of a^2, a^2b^3 and a^2b^2c.

2. $x^3 + b^3$ is the lowest common multiple of $x + b$ and $x^2 - bx + b^2$.

Remark. — In calculating the lowest common multiple of several expressions note that it must contain as factors every different factor of all the expressions. In case a factor occurs more than once in any of the expressions it is to be taken to the highest power that it occurs in any of the expressions.

Find the lowest common multiple of the following:

1. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.
   (Try $3x - 7$ as factor.)
2. $x^2 + x^4$, $2x^2 - 4x$ and $x^2 + 1$.
3. $a^2 + ab + ac + bc$ and $a^2 + 2ab + b^2$.
4. $a^2 - 15a + 50$, $a^2 + 2a - 35$ and $a^2 - 3a - 70$.
5. $x^2 + 5x + 6$, $x^3 - 19x - 30$ and $x^3 - 7x^2 + 2x + 40$.
6. $a^3 + 6a^2 + 11a + 6$ and $a^4 + a^3 - 4a^2 - 4a$.
7. $8a^2 - 6a - 9$ and $6a^3 - 7a^2 - 7a + 6$.
8. $x^2 - 7xy + 12y^2$, $x^2 - 6xy + 8y^2$ and $x^2 - 5xy + 6y^2$.

5. The **highest common factor** of two or more expressions is that expression of highest degree that will be an exact divisor of the given expressions. Thus, abc is the highest common factor of ab^2c^2, a^2bc, abc^3. The highest common factor must contain as factors all the factors that are common to all the expressions. The highest common factor is usually obtained by factoring each of the given expressions and then taking each factor as many times as it is common to all the expressions. The product of these factors is the highest common factor.

When the expressions are not easily factored a method due to Euclid * is useful. Let A and B be two expressions (or numbers). Suppose (1) $A = qB + R_1$, (2) $B = q_1R_1 + R_2$,
(3) $R_1 = q_2R_2 + R_3$, (4) $R_2 = q_3R_3$.

From these we have, since $R_2 = q_3R_3$, R_3 is the highest factor of R_2, R_1, R_3. Hence $B = R_3(q_1q_2q_3 + q_1 + q_3)$.

$$A = R_3(qq_1q_2q_3 + qq_1 + qq_3 + q_2q_3 + 1).$$

Therefore, the highest common factor of A, B is R_3.

This suggests the following rule:

Divide A by B, call the quotient q and the remainder R_1.

Next, divide B by R_1, call the quotient q_1 and the remainder R_2.

Next, divide R_1 by R_2, call the quotient q_2 and so on, continuing to divide the last divisor by the last remainder, until a

* See Geometry. This method may be omitted if desired.

remainder 0 is obtained. The last divisor used is the highest common factor of A, B.

If it becomes evident during the above process that the final division cannot give a remainder 0, then A, B have no common factor different from unity.

Remark. — During the process outlined above we may, when necessary or convenient, divide or multiply any of the expressions A, B, R_1, R_2, . . . by any number not a common factor of A, B without affecting the highest common factor. If a factor be used which is common to A, B, we must account for it in making up the highest common factor of A, B.

The following example will illustrate the process:

Find the highest common factor of

$$6\,x^3 + 7\,x^2y - 3\,xy^2 \text{ and } 4\,x^3y + 8\,x^2y^2 - 3\,xy^3 - 9\,y^4.$$

First we take x out of the first expression and y from the second. We then have

$$6\,x^2 + 7\,xy - 3\,y^2 \text{ and } 4\,x^3 + 8\,x^2y - 3\,xy^2 - 9\,y^3.$$

Since $4\,x^3$ is not divisible by $6\,x^2$ we multiply the second expression by 3. We have then

$$\begin{array}{rrrrl}
6\,x^2 + 7\,xy - 3\,y^2)\,12\,x^3 & + 24\,x^2y & - 9\,xy^2 & - 27\,y^3 & (2\,x + 5\,y \\
12\,x^3 & + 14\,x^2y & - 6\,xy^2 & & \\
\hline
 & 10\,x^2y & - 3\,xy^2 & - 27\,y^3 & \\
 & & & 3 & \\
\hline
 & 30\,x^2y & - 9\,xy^2 & - 81\,y^3 & \\
 & 30\,x^2y & + 35\,xy^2 & - 15\,y^3 & \\
\hline
 & & -44\,xy^2 & - 66\,y^3 &
\end{array}$$

Multiplying by 3,

Divide by $-22\,y^2$, $(-44\,xy^2 - 66\,y^3) \div (-22\,y^2) = 2\,x + 3\,y$.

$$\begin{array}{rrrl}
2\,x + 3\,y)\,6\,x^2 & + 7\,xy & - 3\,y^2 & (3\,x - y \\
6\,x^2 & + 9\,xy & & \\
\hline
 & -2\,xy & - 3\,y^2 & \\
 & -2\,xy & - 3\,y^2 & \\
\hline
\end{array}$$

Therefore $2\,x + 3\,y$ is the highest common factor of the given expressions.

Find the highest common factor of:

1. 16, 72, 144.
2. 1728, 576.
3. $a^2 - 5\,ab + 4\,b^2$ and $a^3 - 5\,a^2b + 4\,b^3$.
4. $2\,a^2 - 5\,a + 2$ and $12\,a^3 - 8\,a^2 - 3\,a + 2$.
5. $x^2 - 1$, $x^3 + 1$ and $x + 1$.
6. $10\,(x + 1)^3$ and $4\,(x + 1)^2\,(x - 1)$.
7. $x^4 - 3\,x^2 - 28$, $x^4 - 16$ and $x^3 + x^2 + 4\,x + 4$.

6. All the preceding processes may be used in the **reduction** and **manipulation of fractional expressions.**

1. $(a^2 - 8\,ab + 7\,b^2)/(a^2 - 3\,ab + 2\,b^2)$ to lowest terms.
2. $(x^6 - y^6)\,(x - y)/((x^3 - y^3)\,(x^4 - y^4))$ to lowest terms.
3. $(a + b)\,((a + b)^2 - c^2)/(4\,a^2c^2 - (a^2 - b^2 - c^2)^2)$ to lowest terms.
4. $(3-x)/(1-3\,x) - (3+x)/(1+3\,x) - (1-16\,x)/(9\,x^2-1)$, combine and reduce.
5. $x/(x - 2\,y) - y/(2\,y - x) - (x - y)^2/(x^2 - 4\,y^2)$, combine and reduce.
6. $x/(x + 2) - 3/(x - 4) + 3/(x - 6) - 1/(x - 8)$, combine and reduce.
7. $(x^2 - yz)/((x + y)\,(x + z)) + (y^2 - zx)/((y + z)\,(y + x)) + (z^2 - xy)/((z + x)\,(z + y))$, combine and reduce.
8. $1/(1 - x) - 1/(1 + x)/1/(1 - x^2) - 1/(1 + x^2)$, combine and reduce.
9. $(2 - x - (6\,x - 11)/(x+4))/(x + 3 - (3\,x - 17)/(x+4))$, combine and reduce.
10. $(27\,x^3 + 1)/(25\,x^2 - 4)/(15\,x^2 - x - 2)/(25\,x - 20\,x + 4)$, combine and reduce.
11. $$\cfrac{1}{3 + \cfrac{1\cdot 3}{4 + \cfrac{3\cdot 5}{4 + \cfrac{5\cdot 7}{4}}}}$$ (Begin at the bottom; combine and reduce.)
12. $$a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a}}}}$$

7. To **solve an equation** is to find a value of some letter in the equation which will, when substituted for that letter, verify or satisfy the equation. The value of a letter (unknown quantity) in an equation which will satisfy the equation is called a **root** of the equation. Then, to solve an equation is to find its root or roots.

It is assumed that the student knows how to solve simple equations, including equations containing fractions. The following exercises will afford review and extension of that knowledge.

1. $(6x + 1)/15 - (2x - 4)/(7x - 16) = (2x - 1)/5.$

Hint. — Multiply each member by the lowest common multiple of the denominators and solve. That is, clear of fractions and solve the equations for x.

2. $1/(x - 2) - 1/(x - 3) = 1/(x - 4) - 1/(x - 5).$
3. $(x + 2)/(x - 3) + (x - 3)/(x + 4) - (x + 4)/(x + 2) = 3.$
4. $(2x - 1)/(2x - 3) - (x^2 - x)/(x^2 + 4) = 2.$
5. $(x - 1)/(x - 2) + (x - 5)/(x - 4) + (x - 7)/(x - 6) + (x - 7)/(x - 8) = 4.$
6. $a/(x - a) - b/(x - b) = (a - b)/(x - c)$, where a, b, c are regarded as known.
7. $bx/a - (a^2 + b^2)/a^2 = a^2/b^2 - x(a - b)/b.$
8. $(x^3 + 1)/(x + 1) - (x^3 - 1)/(x - 1) = 20.$

Note. — It is better in this case to reduce the fractions to lowest terms before solving.

9. $(ax - a^2)/(x - b) + (bx - b^2)/(x - a) = a + b.$
10. $(m + n)/(x + m - n) - 2m/(x - m + n) + (m - n)/(x - m - n) = 0.$

8. A **single equation** is sufficient to determine the value of **one unknown** symbol in the equation. We are to think of an equation containing one unknown as a condition to be satisfied by the unknown. We may impose one independent or arbitrary assumption on one unknown. This assumption may be put in the form of an equation. It is then a problem, more or

less difficult, to determine the value of the unknown that will satisfy the condition. That is, we must find a root of the equation, or, as we say, solve the equation for the unknown.

We may make **two independent assumptions** or impose two independent conditions on **two unknowns simultaneously.** These conditions may be put in the form of equations. We must then solve the equations to determine the unknowns. To do this we must derive from the two equations a single equation with only one unknown. This equation will when solved give the value of one unknown. By substituting this value for that unknown in one of the original equations there will result an equation with the other unknown which may now be determined. The process of deriving a single equation with one unknown from **two** equations each containing **two** unknowns is called **elimination** of an unknown or symbol.

When **three equations** each containing **three unknowns** are given for solution, we must derive **two equations** each containing the same two unknowns and from these finally derive a **single equation** with only one unknown. The last equation being solved gives the value of one unknown. This value when substituted in one of the **two** equations will give an equation from which another unknown can be determined. Having the values of two unknowns they may be substituted in one of the original equations and then from the resulting equation the value of the third unknown can be determined.

In general, the process of deriving $\boldsymbol{n-1}$ **equations** with $\boldsymbol{n-1}$ **unknowns** from n equations with n unknowns is called **elimination.** The idea of elimination and substitution is very important. Elimination is carried out in elementary algebra by several methods. Experience only can teach the pupil the best method to use in a given case. The following three methods are the ones most used:

1. **Combine two equations** by addition or subtraction so as to cause one unknown to disappear. Often one or both equations must be multiplied or divided by some known number in order that addition or subtraction will cause an unknown to disappear.

2. **Solve one of two equations for one unknown** in terms of the other unknowns and substitute this value in the other equations. The resulting equations will contain one less unknown than before.

3. **Solve each of two equations for the same unknown** in terms of the other unknowns and place these two values equal to each other. The resulting equation will contain one less unknown than the two before.

For other methods books on higher algebra or theory of equations must be consulted.

1. Eliminate c from $ay = x - c$, $2\,ay'y = 2\,(x - c)$.

2. Eliminate c from $ax + by = c$, $a'x + b'y = 3\,c$.

3. Solve for x, y from $ax + by = c$, $a'x + b'y = c'$ and save the result as a formula for solving Ex. 4.

4. Apply the results of Ex. 3, to solve $2\,x + 3\,y = 8$, $5\,x - 2\,y = 3$.

5. Solve $10/x - 9/y = 8$, $8/x + 15/y = 1$.

Note. — Do not clear of fractions, but consider $1/x$, $1/y$ as unknowns.

6. Solve $6\,x - 4\,y - 7\,z = 17$, $9\,x - 7\,y - 16\,z = 29$, $10\,x - 5\,y - 3\,z = 23$.

7. If $I = Prt$ and $A = P\,(1 + rt)$ and if $A = 1250$, $r = 0.06$, $I = 250$, find P, t.

8. Solve $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ and save the result as a formula to solve Ex. 9.

9. Apply the result of Ex. 8, to solve $3\,x - 2\,y + z = 5$, $2x + 3\,y - 4\,z = 8$, $x - 5\,y + 3\,z = 6$.

10. Eliminate x from the equations $xy = 12$, $x - y = 1$.

11. If $l = a + (n - 1)\,d$, $S = (a + l)\,n/2$, and if $a = 4$, $S = 60$, $l = 16$, find n and d.

12. If $x = 19$, $y = 1900$; $x = 25$, $y = 3230$; $x = 44$, $y = 9780$ satisfy the equation $y = ax^2 + bx + c$, determine a, b, c.

13. If a straight line passes through the points whose coordinates are $x = 2$, $y = 3$; $x = -4$, $y = 6$ and if the equation of the line is $y = mx + b$, find m, b.

9. Formulas relating to radicals: All ordinary operations with radicals are based on the following:

1. $a^{\frac{1}{n}} = \sqrt[n]{a}$ *
2. $a^{p/q} = \sqrt[q]{a^p}$.
3. $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$.
4. $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$.
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.
6. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.
7. $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$.
8. $b\sqrt[n]{a} = \sqrt[n]{ab^n}$.
9. $\sqrt[n]{a^m} \cdot \sqrt[n]{a^{n-m}} = a$.
10. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.

These formulas are extensions of the index laws. By means of formula 1 and the index laws in article 2 the remaining nine formulas are easily verified.

Simplify the following:

1. $\sqrt{121}$. Thus, $\sqrt{121} = \sqrt{11^2} = 11$. See formula (1).
2. $\sqrt[12]{625\,x^{12}y^4z^8}$. Thus, $\sqrt[12]{625\,x^{12}y^4z^8} = \sqrt[3]{5\,x^3yz^2} = x\sqrt[3]{5\,yz^2}$.
3. $\sqrt{480}$; $\sqrt{\frac{7}{8}}$ (use formulas 8, 9). Thus, $\sqrt{\frac{7}{8}} = \sqrt{\frac{7}{8} \times \frac{2}{2}}$ $= \sqrt{\frac{14}{16}} = \frac{1}{4}\sqrt{14}$.
4. $\sqrt[3]{81\,a^4/16\,bc^2}$.
5. $\sqrt[3]{320} - \sqrt[3]{135} + \sqrt[3]{625}$.
6. $\sqrt{\frac{25}{18}} + \sqrt{\frac{49}{27}} + \sqrt{\frac{5}{6}}$.

Reduce to common index:

7. $\sqrt{3}, \sqrt[4]{5}, \sqrt[6]{15}$.

Use formula (3), reduce exponents to common denominator and pass back to radicals.

* If a is negative and n an even integer, $\sqrt[n]{a}$ is said to be an imaginary quantity. Such quantities are written as follows:

$$\sqrt{-x} = \sqrt{-1}\,\sqrt{x} = i\,\sqrt{x}$$

where x is positive.

8. $\sqrt{12} + \sqrt{75} + \sqrt{147} - \sqrt{48}$.

9. $\sqrt[3]{2a^2b}$, $\sqrt[6]{6b^2c^5}$, $\sqrt[9]{14c^4a^7}$, to common index.

10. $\sqrt{12} + \sqrt{3} - \sqrt{45}$.

11. $\sqrt[3]{3}$, $\sqrt[6]{5}$, $\sqrt[8]{11}$, to common index.

12. $\sqrt{50} - \sqrt{4\frac{1}{2}} - \sqrt[3]{-24} - \sqrt[3]{7\frac{1}{9}}$.

13. $\sqrt[3]{4ab^2} \cdot \sqrt[5]{8b^4c^3}$ (to common index before multiplying).

14. $\sqrt{24} \cdot \sqrt[6]{\frac{9}{8}} \cdot \sqrt[8]{\frac{8}{27}}$.

15. $(\sqrt{6} + \sqrt{10} + \sqrt{14}) \div \sqrt{2}$.

16. $\sqrt{5 - 2\sqrt{2}} \cdot \sqrt{5 - 2\sqrt{2}}$.

17. $(x + y)\sqrt{(x - y)/x + y} - (x - y)\sqrt{(x + y)/(x - y)} + \sqrt{1/(x-y)}$.

18. $(\sqrt{3} - \sqrt{2})/(\sqrt{2} - \sqrt{5})$, use formula (10) to remove radicals from the denominator.

19. $(a + \sqrt{b})/(a - \sqrt{b})$.

20. $(3 + \sqrt{6})/(\sqrt{2} + \sqrt{3})$.

Solve the following equations and substitute the results in the original equation in each case.

21. $x^{\frac{1}{4}} = 4$, raise both members to the fourth power.

22. $(\sqrt{2x - 1})^{\frac{1}{3}} = \sqrt[3]{3}$, raise both members to the sixth power.

23. $\sqrt{x - 4} + \sqrt{x - 11} = 7$, square both members, transpose, square again.

24. $1/(\sqrt{x + 1}) - 1/(\sqrt{x - 1}) + 1/\sqrt{x^2 - 1} = 0$, clear of fractions and proceed as in (23).

25. $\sqrt{2 + \sqrt{3 + \sqrt{x}}} = 2$.

26. $x - 7 - \sqrt{x^2 - 5} = 0$, transpose $x - 7$ before squaring.

27. $x^2 - 7 - \sqrt{x^2 - 4} = 0$, x^2 is to be considered the unknown at first.

10. Each student should be able to solve quadratic equations quickly and accurately by some method. The theory of equations of the second degree and of higher degrees will be

treated in a later chapter. Here will be given a formula which will answer all present needs. Suppose the quadratic equation has been reduced to the form

$$ax^2 + bx + c = 0.$$

Then the two roots are given by the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}.$$

A **pair of simultaneous equations** with two unknowns, where one equation is of the **second degree** and one of the **first degree,** may be solved by the second method of elimination, **8.** Solve the first degree equation for one of the unknowns in terms of the other and substitute in the second degree equation. The resulting equation will be a quadratic with one unknown which can be solved by the above formula. The other unknown can then be found by substituting in the first degree equation.

Solve the following equations:

1. $x^2 - 2\,x = 35.$
2. $x^2 - 3\,x - 1 - \sqrt{3} = 0.$
3. $2\,x/(x+2) - (x+2)/2\,x = 2.$
4. $15\,x^2 - 86\,x - 64 = 0.$
5. $4\,x^4 - 17\,x^2 - 18 = 0$, consider x^2 as the unknown at first.
6. $3\,x^{\frac{3}{2}} - 4\,x^{\frac{3}{4}} = 7.$
7. $x^2 - 1/x^2 = a^2 - 1/a^2.$
8. $\sqrt{3-x} + \sqrt{2-x} = \sqrt{5 - 2\,x}.$
9. $(x^3+3x+1)/(4x^2+6x-1) = 3\,(4x^2+6x-1)/(x^2+3x+1).$
10. $\begin{cases} 7\,x^2 - 6\,xy = 8, \\ 2\,x - 3\,y = 5. \end{cases}$
11. $\begin{cases} 1/x^3 + 1/y^3 = 1001/125. \\ 1/x + 1/y = 11/5. \end{cases}$ Divide first equation by second member by member and solve resulting equation with second.
12. $\begin{cases} 3\,(x^3 - y^3) = 12\,xy. \\ x - y = 1. \end{cases}$
13. $\begin{cases} x - y = a. \\ xy = (a^2 - b^2)/4. \end{cases}$

11. Inequalities. — Often the most important fact about two numbers is that one is greater than or less than the other. To express such relations briefly a sign of inequality is used as follows: $a > b$ means a is greater algebraically than b, or that b is less than a.

If $b < a$ then $-b \nless -a$ ($\nless$ means not less than), for we know that $-b > -a$ if $a > b$.

The following laws hold for inequalities as for equations:

1. Both sides of an inequality may be multiplied or divided by the same positive number without affecting the sense of the inequality.

2. Equal numbers may be added to or subtracted from both sides of an inequality. A term may be transposed.

If both sides of an inequality be multiplied or divided by the same negative number the ense of the inequality is reversed and the vertex of the sign must be pointed in the opposite direction.

1. For what values of x is the expression $7x - 23/3 < 2x/3 + 5$? Multiply both sides by 3, $21x - 23 < 2x + 15$.

Transpose, $19x < 38$.

Divide by 19. $x < 2$.

Therefore the inequality holds for all values of x less than 2.

2. For what values of x is

$$(x-1)(x-2)(x-3) < (x-5)(x-6)(x-7).$$

3. Show that for all values of x, $9x^2 + 25 \nless 30x$.

4. For what values of x is $4x^2 - 4x - 3 < 0$.

First determine for what values of x the expression is 0. Then by inspection determine the ones that satisfy the inequality.

5. Show $a/b + b/a > 2$, for all positive values of a, b and $b \neq a$.

6. Show $(a+b)(a^3+b^3) > (a^2-b^2)^2$.

7. For what values of x is $x - 7 > 3x/2 - 8$.

8. For what values of x is $(x-1)(x-3)(x-6) > 0$.

By choosing values of x and determining the signs of the factors the answer can be deduced without calculating.

12. The **binomial formula** is of frequent use in mathematics. Let us assume:

$$(1)\quad (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)\ a^{n-2}b^2}{1\cdot 2} + \cdots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+2)\,a^{n-r+1}b^{r-1}}{1\cdot 2\cdot 3\ \ldots\ (r-1)}$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)\,a^{n-r}b^r}{1\cdot 2\cdot 3\ \ldots\ (r-1)\,r}$$
$$+\cdots+b^n.$$

Multiplying both sides by $a+b$ and carrying out the actual multiplication,

$$(2)\quad (a+b)^{n+1} = a^{n+1} + (n+1)\,a^n b + \frac{(n+1)\,na^{n-1}b^2}{1\cdot 2} + \cdots$$
$$+\frac{(n+1)n(n-1)\ \ldots(n-r+3)\,a^{n-r+2}b^{r-1}}{1\cdot 2\ \ldots\ (r-1)}$$
$$+\frac{(n+1)n(n-1)\ \ldots(n-r+2)\,a^{n-r+1}b^r}{1\cdot 2\ \ldots\ r}+\ldots+b^{n+1}.$$

It is seen that equation (2) is exactly what equation (1) would become if in the latter $n+1$ is put in place of n. By actual trial it is known that (1) holds for $n = 2$. Then (2) gives immediately the value of $(a+b)^3$. Now in (1) put $n = 3$ and (2) gives the value of $(a+b)^4$. This process may be continued indefinitely. It follows that (1) and (2) hold for any and all positive integral values of n. It will be assumed here and proved later that (1), (2) hold for fractional and negative values of n.

1. Find without multiplying the 5th power of $1 - x$.
2. Find without multiplying the $\frac{1}{2}$ power of $1 - x$ to 4 terms.
3. Find without multiplying the $-\frac{1}{2}$ power of $1 - x$ to 4 terms.
4. Find without multiplying the -1 power of $1 - x$ to 4 terms.
5. Find without multiplying the $\frac{1}{3}$ power of $1 - x$ to 4 terms.

CHAPTER II

GEOMETRICAL THEOREMS AND FORMULAS *

13. 1. A circle of given radius can be described about any point as a center.

2. In the same, or in equal circles, equal central angles intercept equal arcs.

3. The perimeters of inscribed polygons are less than the circle and the perimeters of circumscribed polygons are greater than the circle.

4. If two straight lines intersect, the vertical angles formed are equal.

5. If two parallel lines are cut by a transversal, the alternate interior angles formed are equal.

6. If two parallel lines are cut by a transversal, the corresponding angles formed are equal.

7. Angles having their sides parallel each to each and extending in the same direction or in opposite directions from the vertex are equal. If one pair of parallel sides extend in the same direction and the other pair in the opposite direction the angles are supplementary.

8. Angles having their sides perpendicular each to each, and both acute or both obtuse, are equal.

9. The sum of all the angles of a triangle is equal to a straight angle.

10. In any triangle there must be at least two acute angles.

11. In any isosceles triangle, the angles opposite the equal sides are equal.

12. An equilateral triangle is equiangular.

13. In an isosceles triangle the bisector of the vertical angle is the perpendicular bisector of the base.

* These are given chiefly for reference use.

14. If two sides and the included angle of one triangle are equal to the corresponding parts of another, the triangles are congruent.

15. If two angles and a side of one triangle are equal to the same parts of another, the triangles are congruent.

16. If three sides of a triangle are equal to three sides of another, the triangles are congruent.

17. Two lines perpendicular to two intersecting lines, respectively, must meet.

18. If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

19. The diagonals of a parallelogram bisect each other.

20. If the diagonals of a parallelogram are equal, the figure is a rectangle.

21. The diagonals of a rhombus bisect each other at right angles.

22. The sum of the interior angles of a convex polygon of n sides is $n - 2$ straight angles.

23. The sum of the exterior angles of a convex polygon is two straight angles.

24. Every point on the perpendicular bisector of a sect is equidistant from the ends of the sect. (A sect is a segment of a straight line.)

25. Any straight line parallel to a side of a triangle forms with the other two sides a triangle similar to the first. (The two sides produced if necessary.)

26. A line parallel to one side of a triangle divides the other two sides in the same ratio.

27. Homologous sides of similar triangles have the same ratio.

28. In any two similar polygons (1) the homologous angles are equal; (2) the homologous sides are proportional.

29. In any two similar polygons the ratio of any two homologous lines is equal to the ratio of any other homologous lines and equal to the ratio of the perimeters of the polygons.

30. The longer side of a triangle is opposite the greater angle.

31. In any triangle any side is less than the sum of the other two sides and greater than their difference.

32. The diameter of a circle is twice its radius.

33. A circle is determined by (*a*) the center and radius; (*b*) center and diameter; (*c*) three points not in a straight line.

34. A perpendicular bisector of a chord passes through the center of the circle.

35. A straight line perpendicular to a radius at its extremity is tangent to the circle.

36. Equal chords are equally distant from the center of the circle.

37. Two angles at the center have the same ratio as their intercepted arcs.

38. The area of a rectangle or a parallelogram equals the product of the base by the altitude.

39. The area of a triangle equals the product of the base by half the altitude.

40. The area of a circle equals π times the square of its radius, (πr^2). $\pi = 3.1416$ approximately.

41. The area of a sphere equals 4π times the square of its radius, ($4\pi r^2$).

42. The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.

43. The altitude of a right triangle from the right angle is a mean proportional between the segments into which it divides the hypotenuse.

44. The volume of a prism or cylinder equals the area of its base times its altitude.

45. The areas of similar polygons have the same ratio as the squares on any two homologous lines.

46. Area of triangle whose sides are a, b, c, is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

when $2s = a + b + c$.

47. Area of sector of circle $A = \dfrac{arc \times r}{2}$.

48. Area of segment of sphere $A = 2\pi rh$ where h is altitude of segment.

49. Area of curved surface of cylinder, $A = 2\pi rh$.

50. Area of curved surface of cone, $A = 2\pi r\dfrac{s}{2}$, where s is slant height.

51. Area of a segment (of one base) of a circle is the area of the sector subtended by the arc of the segment minus the area of the triangle whose base is the chord of the segment and vertex the center of the circle.

CHAPTER III

METHODS OF CALCULATION

14. The **need of numbers** and number calculations arose quite naturally in race development. Recognition of number and geometric form appears not to be peculiar to the human race. There is good reason for believing that many of the lower races of animals have the ability to count and to recognize space forms and magnitudes. A few of the conditions in human affairs that called forth methods of counting, measuring and reckoning are enumerated below.

1. Numbering and comparing groups of objects.
2. Barter and trade of primitive commerce.
3. Census and tribal statistics.
4. Calendar and time measurements.
5. Land measurements.
6. Determination of weights and measures.
7. Taxation.
8. Modern commerce.
9. Engineering and exact science.
10. Personal and corporation accounting.
11. Insurance, savings accounts and investments.
12. Statistical and social science.

15. Important discoveries in the art of calculating. — (*a*) **Hindu notation:** We owe our present method of writing ordinary numbers to the Hindus. The important principles of this notation and the ones that make it superior to all others, to date, are its position value and zero. This notation probably antedates the Hindus but it was through them that it was transmitted to the western nations.

(*b*) **Decimal fractions** were given to the world about the end of the sixteenth century by Simon Stevins of Bruges in Belgium. Fractions had been a hard problem for the race. For a long time all fractions were expressed as the sums of unit fractions. Thus 7/12 was regarded as 1/2 + 1/12 and similarly for other fractions. Other methods were developed gradually. The discovery that fractions could be incorporated into the number system in decimal form as an extension of the Hindu notation was a real contribution not only to mathematics but to civilization.

(*c*) Early in the seventeenth century the **discovery of logarithms** gave the computer and mathematician another powerful instrument. It was the crowning achievement in numbers. Coming at a time when Kepler was working on his planetary theory and immediately following the computation of trigonometric tables by the Germans, the discovery of logarithms was of great value. The names of Napier, the inventor, and Briggs, the collaborator and editor, of the tables have been made immortal by these contributions. Laplace remarked that the discovery of logarithms would double the life of the astronomer by shortening the labor of his calculations.

16. Many **mechanical contrivances** have been and are still in use for shortening the labor of computing. In passing we may mention: (*a*) The **abacus,** an ancient instrument, still in use in some countries and used in our schools under the name, numeral frame; (*b*) the **slide rule,** which is based on the idea of logarithms. It is convenient and adequate for many purposes in applied science and in commerce; (*c*) the **calculating machine,** such as the Burroughs, does all the ordinary arithmetical calculations with a speed and accuracy that justifies its use in banks and mercantile houses where much computing is done; (*d*) geometrical methods of performing the arithmetical operations were in possession of the Greeks. In certain branches of engineering, geometric methods are now used with great efficiency. They depend on the principle of the proportionality of the sides of similar triangles. (*e*) General reckoning tables,

containing products, quotients, powers, roots and reciprocals of numbers may be obtained by those who desire to use them.

17. Graphic or geometric representation of numbers. — A number is graphically represented by a straight line of length proportional to the magnitude of the number. The factor of proportionality determines the scale of the representation. Thus, if a line 1″ * long is to represent the number 50, the scale is 50 to 1 in inches. If the number 275 is to be represented by a line not over 4″ long, we divide 275 by 4 and obtain 69, nearly. It would, therefore, be safe to use a scale of 70 to 1 or any larger number than 70 to 1.

It must be remembered that the smaller the scale (larger ratio of proportionality), the more difficult it is to estimate small numbers or odd units. On the other hand the larger the scale (smaller ratio of proportionality), the larger the drawing and the larger the paper or other surface required. The scale is to be determined by convenience and by the degree of accuracy demanded. Very large drawings are sometimes necessary. Sometimes very small drawings will do.

1. (*a*) Lay off 350 to the scale of 50 to 1″; (*b*) lay off 350 to the scale of 100 to 1″.

2. Measure the lines below to the scale of 50 to 1″ and determine the numbers they represent.

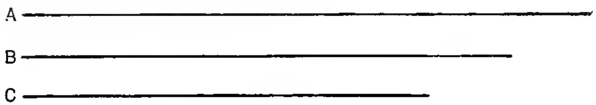


FIG. 1.

18. Arithmetical operations. — (1) To **add** a number b to a number a: Let $O1$ be the unit length. Suppose $a = OA$, $b = AB$. Obviously $OB = OA + AB$ represents the sum of a and b, to the unit $O1$.

FIG. 2.

* The notation, 1″, means 1 inch; 1′ means 1 foot.

To subtract b from a lay off b from A toward B'. Then $a - b = OA - AB = OB'$.

(2) **To multiply** a number a by a number b. Let $O1$ be the unit. Lay off $a = OA$. From A draw a line AC making any convenient angle with OA. Lay off $1B$ parallel to AC and equal to b. Draw OB and produce it until it intersects AC at C, say. Then AC represents the product ab. That is, $c = ab$. For, by the similar triangles OAC, $O1B$,

$$O1 : 1B :: OA : AC$$

or

$$O1 \cdot AC = 1B \cdot OA$$

and

$$AC = 1B \cdot OA,$$
$$c = ba,$$

since $O1 = 1$, the unit of measure.

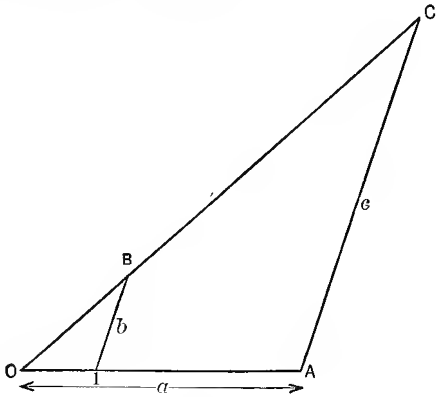


FIG. 3.

(3) **To divide** a number c by a number a. In this problem use the diagram of the preceding problem and solve the last equation for b. It is seen that if a is the divisor and c the dividend, the quotient is b.

The **reciprocal** of a number can be found as a special case where the dividend is 1. In using this method for finding

reciprocals, it is advisable to employ a larger scale for the dividend than for the divisor.

The **square** of a number can be found as a special case of problem (2) where $a = b$.

(4) To find the **square root** of a number a: Evidently $a = a \cdot 1$ and $\sqrt{a} = \sqrt{a \cdot 1}$. Make $BC = a$ and $AB = 1$. Draw on AC as a diameter, a circumference, AKC. At B erect a perpendicular meeting the circumference at D. Then $\overline{BD}^2 = AB \cdot BC$ by geometry. Hence $\sqrt{a} = BD$.

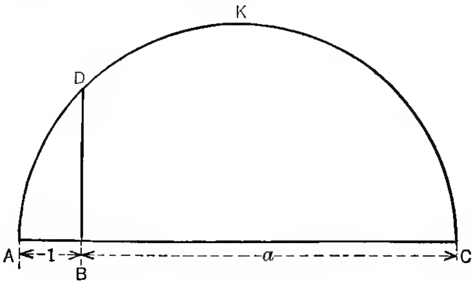


FIG. 4.

1. Find the product of 35×73; 18×93; 36×13 by the geometric method.

2. Find the quotient of $125/16$; $39/4$; $50/13$ by the geometric method.

3. Find the reciprocals of all integers from 2 to 10, inclusive, by the geometric method.

4. By the geometric method find $\sqrt{12}$; $\sqrt{27}$; $\sqrt{18}$; $\sqrt{56}$.

5. By the geometric method find $(35 \times 12)/16$; $18 \times 27/13$.

19. Idea of logarithms. — The student should now recall the index laws, see **2**, Chap. I.

All positive numbers can be regarded as powers of some positive number, not unity. It is, of course, true that most such powers cannot be exactly expressed, but close approximations can be determined and exactly expressed. This is because these indices are not rational numbers. A rational number can be expressed as the quotient of two integers.

What power of 2 is 8?
What power of 2 is 10?
What power of 10 is 10?
What power of 10 is 100?
What power of 10 is 1/10?

20. Definition of the logarithm of a number. — If $y = a^x$, x is called the logarithm of y to the base a. Briefly we may write $x = \log_a y$. $y = a^x$ implies $x = \log_a y$ and conversely.

From this definition it follows that

$$\log_{10} 10 = 1, \text{ since } 10 = 10^1,$$
$$\log_{10} 100 = 2, \text{ since } 100 = 10^2,$$
$$\log_{10} 1/100 = -2, \text{ since } 1/100 = 10^{-2}.$$

21. Rules for calculating with logarithms are derived directly from the index laws. Thus,

$$10^2 \cdot 10^1 = 10^{2+1} = 10^3 = 1000.$$

That is:

(1) $\log_{10}(10^2 \cdot 10) = \log_{10} 1000 = 3 = 2 + 1 = \log_{10} 100 + \log_{10} 10.$

Again:

(2) $\log_{10}(1000 \div 10) = \log_{10} 100 = 2 = 3 - 1 = \log_{10} 1000 - \log_{10} 10.$

Generalized, (1) and (2) give the rules for multiplication and division by means of logarithms. Let x and y be any two positive numbers and p their product. Then,

(1′) $$\log_a p = \log_a x + \log_a y.$$

If q is the quotient x/y, then

(2′) $$\log_a q = \log_a x - \log_a y.$$

Consider (3) $y = x^n$, where n is any number. Then, by **20**

$\log_a y = \log_a y$ or $y = a^{\log_a y}$, $x = a^{\log_a x}$, $x^n = (a^{\log_a x})^n = a^{n \log_a x}$.

Therefore,

(3′) $$\log_a y = \log_a x^n = n \log_a x.$$

This result shows how to use logarithms to raise a number to any power.

Now consider:

$$y = \sqrt[m]{x} = x^{\frac{1}{m}}. \tag{4}$$

We can write:

$$\log_a y = \log_a x^{\frac{1}{m}} = \frac{1}{m} \log_a x. \tag{4'}$$

This result shows how to use logarithms to find the roots of numbers.

The foregoing equations (1′), (2′), (3′), (4′) may now be translated into rules.

1. The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.

2. The logarithm of the quotient of one number divided by another is the logarithm of the dividend minus the logarithm of the divisor.

3. The logarithm of the nth power of a number is n times the logarithm of the number.

4. The logarithm of the nth root of a number is the logarithm of the number divided by n.

5. The logarithm of the reciprocal of a number is the negative of the logarithm of the number. (See co-logarithm below.)

It may be more convenient to use addition when dividing with logarithms. To do this the logarithm may be subtracted from 0 or $10 - 10$, or $20 - 20$, etc. Thus instead of subtracting $\log n = 3.37581$, it may be more convenient to use addition. This is easily done by first subtracting 3.37581 from $10 - 10$. Thus we may add $6.62419 - 10$. The advantage of this method lies in the fact that it is easy to make the above subtraction directly as the logarithm is taken from the table by subtracting each figure, beginning at the left, from 9 and the last figure on the right from 10. The result of this subtraction is called the **co-logarithm** of the number or arithmetic complement of the logarithm.

22. From **19** and **20** it is evident that when 10 is the base of the logarithms, the following statements hold:

1. Any number between 1 and 10 has a proper fraction for its logarithm.

2. Any number between 10 and 100 has 1 plus a proper fraction for its logarithm.

3. Any number between 100 and 1000 has 2 plus a proper fraction for its logarithm.

4. Any number between 0 and 1 has a negative logarithm.

It is seen that logarithms of numbers consist of two parts, an integer or zero and a fraction. The integer part is called the **characteristic.** The fractional part is called the **mantissa** of the logarithm.

For convenience in calculating, a logarithm should be written so the part on left of decimal point is positive. The mantissa is always positive. Thus, $\bar{2}.34567$ (where $\bar{2}$ shows that the characteristic, only, is negative) may be expressed as $8.34567 - 10$ where the part of the characteristic on left of the decimal point is positive.

Consider:

$$\begin{aligned}
\log 273 &= 2 \text{ plus a proper fraction.}\\
\log 27.3 &= 1 \text{ plus the same proper fraction.}\\
\log 2.73 &= 0 \text{ plus the same proper fraction.}\\
\log 0.273 &= \bar{1} \text{ plus the same proper fraction.}\\
\log 0.0273 &= \bar{2} \text{ plus the same proper fraction.}
\end{aligned}$$

These results are evident from the fact that, when the decimal point is moved one place to the left, the number is divided by 10 and exactly 1 is subtracted from its logarithm, the mantissa remaining the same. It is easy to see that the characteristic depends on the position of the decimal point in the number, while the mantissa depends on the sequence of digits in the number. It will be noticed that with a number $\geqq 1$ *the characteristic is an integer one less than the number of significant figures of the number to the left of the decimal point.* This rule is general if the number of 0's immediately to the right of the decimal point be considered as a negative number of figures on the left. Note that the above example verifies this statement.

What is the characteristic of the logarithm of each of the following numbers: 3.047, 37.56, 0.000842, 1.0045, 67,543, 0.43?

23. In order to use logarithms in calculating it is necessary to have a table of logarithms of all numbers used in the calculations. The mantissas only are given in the table. The characteristic must be supplied in every case according to the above rule.

To construct a table of logarithms involves great labor. It is beyond the scope of this course to explain the method of calculating such tables. For this information the student is referred to works on higher algebra or calculus. A series from which logarithms may be calculated will be given in Chap. XVI. A four place table of logarithms of numbers with explanations is to be found in the back of the text.

Solve the following by the use of logarithms.

1. $79 \cdot 470 \cdot 0.982$.

By the rule for multiplication of **21** write

$$
\begin{aligned}
\log (79 \cdot 470 \cdot 0.982) &= \log 79 + \log 470 + \log 0.982 \\
&= 1.8976 \\
& + 2.6721 \\
& + 9.9921 - 10 = \bar{1}.9921 \\
&= \overline{14.5618} - 10 = 4.5618 = \log 3646.
\end{aligned}
$$

2. $9503 \cdot 0.7095 = ?$
3. $8075 \div 364.9 = ?$
4. $(-0.643) \cdot 0.7564 = ?$
5. $(2.588)^5 = ?$
6. $(0.57)^{-4} = ?$
7. $\sqrt[3]{-0.3089} = ?$
8. $(0.000684)^{\frac{4}{3}} = ?$
9. $\sqrt{943 \cdot (-7298)/(0.00006 \cdot (-99))} = ?$
10. $(\sqrt[7]{0.0476} \cdot \sqrt[5]{222})/\sqrt[3]{5059 \cdot 0.0884} = ?$

24. An **exponential equation** is one in which the unknown quantity occurs as the exponent of some known number in the equation. Such equations can often be easily solved by the use of logarithms.

Find the value of x in the equation $13^{2x} = 14^{x+1}$. Taking logarithms of both sides,

$$2\,x \log 13 = (x + 1) \log 14$$

and

$$x = \frac{\log 14}{2 \log 13 - \log 14}.$$

Substituting the logarithms of 13 and 14 gives $x = 1.06$.

Use logarithms in solving the following problems.

1. Given $A = P\,(1.0\,r)^t$, $I = A - P$, find the compound interest of \$250 at 5 per cent for 25 years. (P = principal, r = rate, t = time, A = amount.)

Note. — $.0\,r$ may be written $r/100$ if preferred.

2. In how many years will \$5000 amount to \$6000, compounded annually at 6 per cent? Given $A = P\,(1.0\,r)^t$.

3. Find the value of x in $5^{x+5} = 8^{x+1}$.

4. Find the value of k in $P_z = P_0 e^{-kz}$, if $P_0 = 14.72$, $z = 1122$, $P_z = 14.11$, $e = 2.718$.

5. How many gallons in a cylindrical tank 3′ in diameter and 6′ 8″ high?

Note. — 6′ = 6 ft.; 8″ = 8 in. This notation will be used from here on.

6. If $c = (Ld/Rb)\,(26^2 - n^2)$, find c when $L = 1650$, $b = 500$, $n = 25$, $R = 1000$, $d = 0.75$.

7. Compute the simple interest of \$135.70 at $3\frac{3}{4}$ per cent for $12\frac{1}{2}$ yrs. ($I = Prt$.)

8. In the triangle ABC, $a = 175$, $b = 225$, $c = 190$. Find the area. ($A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ and $s = (a + b + c)/2$.

9. Find the cost of covering a floor like the figure at \$1.75 per sq. yd.

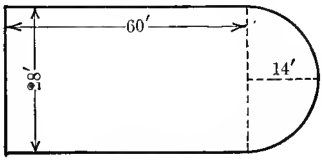


Fig. 5.

10. Find the number of gallons of paint required to paint a cylindrical tank 10′ in diameter and 40′ long (curved surface and both ends to be painted), if 1 gal. paint is sufficient to paint 100 sq. ft. of surface.

11. A grindstone will stand a rim speed of 2500′ per min. How many r.p.m. (revolutions per min.) will a stone 42″ in diameter stand?

12. If the cutting speed of a lathe is 40′ per min., how many r.p.m. must a lathe have to give the desired speed for cutting a piece 2″ in diameter?

13. Find the number of barrels capacity of a rectangular cistern 6′ × 8′ × 10′. (7.48 gal. = 1 cu. ft.)

14. A pie is 10″ in diameter. It is cut in 6 equal sectors about the center. What is the area of the upper surface of each piece?

15. Find the value of a bin of wheat 8′ 8″ × 10′ 6″ × 4′ 3″ at $1.95 per bu.

25. The logarithmic scale. — If distances proportional to the logarithms of numbers are laid off from one end of a straight line segment AB, the segment so divided is a logarithmic scale.

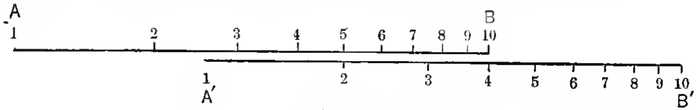


FIG. 6.

If a duplicate scale $A'B'$ is laid along side AB so as to slide parallel to itself, we can use these scales to perform the operations of multiplication and division of numbers. For example, to multiply 25 × 30, set A' at 25 (between 2 and 3) on AB. Run over on $A'B'$ to 30 (at 3 on $A'B'$) and on AB opposite this read 75 (between 7 and 8) on AB. This gives the figures of the product. Knowing the orders of the numbers we know the product is 750.

Since the scales are proportional to the logarithms of numbers, what has been done is to add the logarithm of 30 to the logarithm of 25 to obtain the logarithm of 750. An instrument operating in this way is called a slide rule. Directions for use

generally accompany the rule. The student should now obtain a slide rule and learn to use it as an economy and as a check in his calculations. Commercial and engineering concerns employ the slide rule for certain types of work with great advantage.

26. For convenience of reference, **rules** for the ordinary **Manheim slide rule** are appended here.

Multiplication. — Set the index of C scale over multiplicand on D scale. Under the multiplier on C scale read product on D scale.

Division. — Above dividend on D scale set divisor on C scale. Under index of C scale read quotient on D scale.

Proportion. — Set first term on C scale over second term on D scale. Under third term on C scale read fourth term on D scale.

Squares. — Set the runner on the number on D scale. Read square under runner on A scale.

Square root. — If number has an odd number of integral digits use left half of A scale; if an even number of integral digits use right half of A scale. Set runner over number on A scale. Read square root under runner on D scale.

Cubes. — Set index of B scale under number on A scale. Read cube on A scale over number on C scale.

Cube root. — Under number on A scale move slide until number on A scale above index of B scale is same as appears on C scale under given number on A scale.

Exercise. — Solve all problems of **18** by use of the slide rule.

26*a*. Methods of Interpolation. — Immediately preceding the tables in the back of the text is an explanation of the use of those tables. A number of problems are worked which illustrate simple interpolation.

Below are given problems which illustrate "double interpolation." Other types of interpolation are given in Chapters IV and VIII.

Following is a section of a table given in the U. S. Weather Bureau Bulletin No. 235. A similar table is used in connection with work in artillery and in the navy.

TABLE. — RELATIVE HUMIDITY, PER CENT — FAHRENHEIT TEMPERATURES

Pressure = 29.0 Inches

Air temp. t.	Depression of wet-bulb thermometer $(t-t')$.																			
	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
40°	96	92	88	84	80	76	72	68	64	61	57	53	49	46	42	38	35	31	27	23
41	96	92	88	84	80	77	73	69	65	62	58	54	50	47	43	40	36	33	29	26
42	96	92	88	85	81	77	73	70	66	62	59	55	51	48	45	41	38	34	31	28
43	96	92	88	85	81	78	74	70	67	63	60	56	52	49	46	43	39	36	32	29
44	96	93	89	85	82	78	74	71	68	64	61	57	54	51	47	44	40	37	34	31
45	96	93	89	86	82	79	75	71	68	65	61	58	55	52	48	45	42	39	36	33

Example: Air temperature $t = 42.7°$.
Reading of wet-bulb thermometer $t' = 34.5°$.
Depression of wet-bulb thermometer $(t - t') = 8.2°$.
Determine the relative humidity from above table.

Tabulated values are:

		8.0	8.5
42		41	38
43		43	39

To a difference of 1° in air temperature (43–42) corresponds a difference of 2 per cent in the relative humidity (43–41) when depression is 8.0°. Therefore, a change of 0.7° in air temperature $(42.7 - 42)$ causes a change of 0.7×2 per cent $= 1.4$ per cent in relative humidity. Similarly, for a depression 8.5°, a difference of 1° in air temperature causes a difference of 1 per cent in relative humidity, and, therefore, a change of 0.7° causes a change of 0.7 per cent in the relative humidity. Our table may now be enlarged to

		8.0	8.5
42		41	38
42.7		42.4	38.7
43		43	39

When the depression is 8.2°, the relative humidity by the usual method of interpolation is:

Difference of .5° depression causes a difference of 3.7 per cent in relative humidity (42.4 − 38.7). Therefore, difference of 0.2° depression causes a difference of 1.5 per cent in relative humidity. The required humidity is therefore (42.4 per cent − 1.5 per cent) = 40.9 per cent.

1. Work the above illustrative problem by changing the order of the method, that is, interpolate for temperature 42° and depression 8.2°, which lies between 8.0° and 8.5°. Then do likewise for temperature 43° and so forth.

2. If air temperature, $t = 44.3°$,
   Reading of wet-bulb thermometer $t' = 34.5°$,
   Depression of wet-bulb thermometer $(t - t') = 9.8°$,
   Determine the per cent humidity.

3. Temperature is 41.8°,
   Depression of wet-bulb thermometer $(t - t') = 2.4°$,
   Determine the per cent humidity.

The motion of a projectile is retarded by the resistance of the air. The amount of retardation due to this cause depends upon the temperature and barometric pressure. Following is a portion of a ballistic table from which is determined an "atmospheric factor" that enters into the computation of retardation·

BALLISTIC TABLE

Barometer.	Temperature of air — Fahrenheit degrees.								
	30	31	32	33	34	35	36	37	38
28″	0.994	0.996	0.998	1.000	1.003	1.005	1.007	1.009	1.011
29	0.960	0.962	0.964	0.966	0.968	0.970	0.972	0.974	0.976
30	0.928	0.930	0.932	0.934	0.936	0.938	0.940	0.943	0.945
31	0.898	0.899	0.902	0.903	0.906	0.907	0.909	0.911	0.913

1. Given the temperature $t = 32°$ and the barometric pressure $p = 29''$, then the corresponding atmospheric factor is 0.964. Verify by use of the table.

2. Given the temperature $t = 31.8°$ and the barometric pressure $p = 28.4''$. Find the corresponding atmospheric factor.

Note. — Carry out the interpolations in the same manner as was done in the illustrative problem above on humidity of the air.

3. If $t = 31°$ and $p = 30.8''$, find the corresponding atmospheric factor.

4. If $t = 36.6°$ and $p = 30.2''$, find the atmospheric factor.

SUPPLEMENTARY EXERCISES

1. The diagonal of a square is 73.84′. Find the length of a side.

2. The circumference of a circle is 63.24′. Find the radius. (Use 3.142 for π.)

3. If the area of a circle is given by the formula $A = 0.7854\, d^2$, find the radius of a circle whose area is 842.3 sq. ft.

4. If $1'' = 2.540$ cm., how many centimeters in $8\frac{5}{16}''$?

5. If the barometer reading is 29.92″, what will a barometer read that is graduated in millimeters? If the barometer reading is 75.46 cm., what will a barometer read that is graduated in inches?

6. To find the reciprocal of a number by the use of logarithms: Subtract the mantissa of the log of the number from 1, add 1 to its characteristic and change the sign. The number whose log is this result is the reciprocal of the given number. Use this rule in finding the reciprocal of 543. Find log 1/543 by use of a co-logarithm. How do the two methods for finding the reciprocal of a number differ?

7. Find the reciprocal of 0.00635 by the two methods of the above problem.

8. If 1 cu. ft. of water weighs 62.45 lbs., what is the pressure per sq. in. of a column of water 1 ft. high?

9. The volume of a sphere $v = 4/3\, \pi r^3$. Show that $r = \left(\frac{3\,v}{4\,\pi}\right)^{\frac{1}{3}}$ and find r in cm. when $v = 37.45$ cu. in.

10. The time required for a simple pendulum to make a single oscillation when the angle through which it swings is small, is expressed by $t = \pi\, \sqrt{l/g}$, where l represents the length of the pendulum and g represents the acceleration of gravity. What is the length of a pendulum that vibrates seconds when $g = 979.4$ cm./sec²?

11. The equation of motion of a body falling from rest under the action of gravity is $s = \frac{1}{2}\, gt^2$ where s = distance and t = time.

(a) If $g = 32.16$ ft./sec² and $s = 164.8$ ft., find t.
(b) If $g = 980.3$ cm./sec² and $s = 50.23$ m., find t.

Note that g is expressed in centimeters and s in meters. Must this be taken into consideration when computing t?

12. If a 250 lb. charge of a certain lot of nitro-cellulose powder at 70° F. gives a muzzle velocity of 2275 ft./sec., what should be the weight of a charge to give a muzzle velocity of 2750 ft./sec.? Muzzle velocity and weight of charge are connected by the formula $\frac{V}{V_1} = \left(\frac{w}{w_1}\right)^y$. $y = 1.2$ for nitro-cellulose powder.

13. Using same formula as in problem (12) find the muzzle velocity for a charge of 275 lbs. where the velocity is 2150 ft./sec. at 30° F., for a charge of 240 lbs. of nitro-glycerine powder. $y = 0.8$ for nitro-glycerine powder.

14. The following formulas are used for range correction in gunnery:

$$C_1 = f_w \cdot C, \quad f_w = 1 + \frac{2\,W_x T^{\frac{5}{4}}}{X}.$$

Find C_1 when $T = 36.60$, $W_x = 25.00$, $X = 54{,}000$, and $C = 10.49$.

15. Given $(\frac{1}{4})^n = \frac{1}{2}$; find n.

Note. — Take the reciprocal of each member.

16. Given $(1/2.641)^n = 1/(159.4)$; find n.

17. Given $(1/(0.0649)^n = 1/2.54$; find n.

18. Given $\left(\frac{V_1}{V_2}\right)^{n-1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}$; find n.

Note. — Raise both members of the equation to the power $n/(n-1)$.

19. In problem 18 find n when $V_1 = 0.5$, $V_2 = 6$, $P_1 = 250$ and $P_2 = 7.71$.

CHAPTER IV

GRAPHIC REPRESENTATION

27. Modern **life** and **scientific investigations** make it necessary to use statistics and to draw conclusions from statistical records. These records usually consist of tabulations of measurements in the form of columns of figures. It is not always easy to get all the information contained in a table of statistics without some kind of pictorial aid. When a table of numbers can be put in a form that appeals through the eye to our space notions it is easier to understand and to draw conclusions from such a table.

Various representations to this end are given the name, **graphic representations.** Curves, geometric diagrams, pictures, etc., are frequently used for this purpose. Of the above representations curves are most widely used. In the sequel our study of graphic representation will be limited to the method of curves. The method is best explained by the use of some simple examples.

1. Suppose a man walks for several hours on a winding path and keeps a record of the distance traveled each hour. Let the annexed diagram represent the path and the points reached at the end of each hour.

1st hr., $\frac{1}{2}$ mi.	6th hr., $1\frac{1}{4}$ mi.
2d hr., $1\frac{1}{2}$ mi.	7th hr., $\frac{3}{4}$ mi.
3rd hr., 2 mi.	8th hr., $1\frac{1}{2}$ mi.
4th hr., $\frac{3}{4}$ mi.	9th hr., $\frac{1}{2}$ mi.
5th hr., $\frac{1}{2}$ mi.	10th hr., 2 mi.

To construct the graphic representation of this record proceed as follows: Draw OT and on it, beginning at O, lay off the units

of time to any convenient scale, say $\frac{1}{4}''$ to 1 hr. At O draw OS perpendicular to OT and lay off the units of distance to a scale of, say, $\frac{1}{4}''$ to 1 mi. The point O represents the starting point, 0 mi. and 0 hr. The point 1 of the path is represented at 1_1 on aA, 1 unit to right of OS and $\frac{1}{2}$ unit above OT. The point 2 of the path is represented at 2_1 on bB, 2 units to the right of OS and $\frac{1}{2} + 1\frac{1}{2} = 2$ units above OT. In a similar manner the remaining points of the path are represented on the diagram.

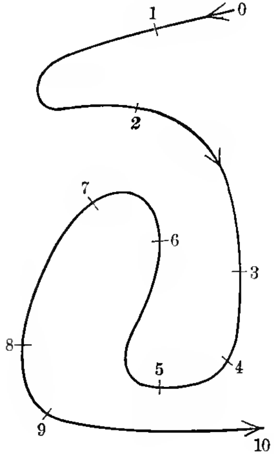


FIG. 7.

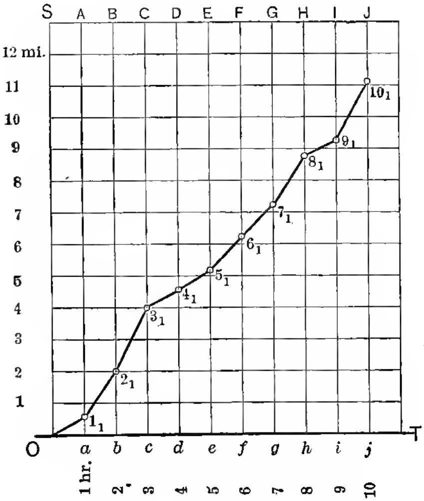


FIG. 8.

The points 0, 1_1, 2_1, . . . , 10_1 are now to be connected by straight line segments.

The broken line $O1_1$, 2_1, . . . 10_1 in a sense represents the travel of the man during a period of 10 hrs. To continue the study of the diagram we need to use certain terms which we shall now define.

28. The line OT is a **base line** and is called the **axis of abscissas.** In this particular case it is a **time axis.** The line OS is a **meridian line** and is called the **axis of ordinates.** In this

particular case it is a **distance axis.** The distance of a point from the meridian is called its **abscissa.** The abscissa must be measured to the same scale as that to which the base line is laid off.

The distance of a point from the base line is its **ordinate.** The ordinate must be measured to the same scale as that to which the meridian is laid off.

The abscissa and ordinate of a point considered together are called its **coördinates.** Evidently the position of a point is determined if its coördinates are known.

The difference of the ordinates of two points divided by the difference of their abscissas, taken in the same order, is called the **slope** of the straight line joining the two points. The slope is also called the pitch of the line.

Resuming the study of the example answer the following questions: (*a*) What does the ordinate of any of the points 1_1, 2_1, . . . , of the broken line represent with reference to the travel of the man? (*b*) What does the abscissa represent? (*c*) What does the slope of any part of the line represent? (*d*) Is the speed of the man constant for the whole time? (*e*) What is the average speed of the man for the whole time? Calculate the average speed for the first 3 hrs. For the second 3 hrs. For the last 3 hrs.

2. A rod 8″ long is set upright on a level table in sunshine. The lengths of the shadow were measured at intervals as follows:

Time of observation = t	Length of shadow = l
10 : 40, A.M.	7.79″
11 : 15, "	7.35″
11 : 45, "	7.25″
12 : 00, M.	7.22″
12 : 10, P.M.	7.22″
1 : 00, "	7.30″
2 : 15, "	7.75″
2 : 30, "	9.81″

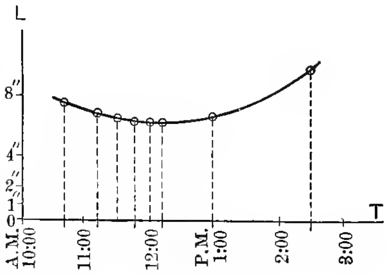


FIG. 9

Construct the graph using time as the abscissa, but draw a smooth curve instead of broken line.*

Take time scale $\frac{1}{2}''$ to one hour, starting at 10:00 A.M. Take length scale $\frac{1}{8}''$ to $1''$. Draw a second curve with same time scale but with length scale $\frac{1}{2}''$ to $1''$. What difference do you notice? Measure the ordinates at 11:00 A.M., and at 1:30 P.M., and determine the probable length of the shadow at these times.

The graphic representation enables us to determine, approximately at least, more values than are given in the measured data. The determination of new values between the old ones is called interpolation. Interpolation is a very useful application of the graphic representation.

3. The population of the U. S. by decades was as follows, in millions:

1790	3.9 millions	1830	12.8 millions	1870	38.5 millions
1800	5.3 "	1840	17.0 "	1880	50.0 "
1810	7.2 "	1850	23.2 "	1890	62.5 "
1820	9.7 "	1860	31.3 "	1900	76.3 "
				1910	92.0 "

(*a*) Construct a curve showing the probable population at any time, using time as abscissa.

(*b*) Connect the decade points of the curve by straight lines and determine the slope of each segment. What information does the slope give?

(*c*) Construct a curve on same abscissas as (*a*) using the slopes found in (*b*) as ordinates. What information does this curve picture?

4. The following data were taken on a certain machine while in operation. The relative velocity of two moving parts was measured in ft. per sec. (ft./sec.), and the coefficient of friction for each velocity was measured.

v = speed, ft./sec. =	1	3	5	7	10	15	20
w = coef. of friction =	0.15	0.122	0.104	0.092	0.079	0.066	0.058

* The curve is an attempt to represent the conditions between observed points, the broken line is not, except in special cases.

Note. — Coefficient of friction is the resistance to sliding of one part on another divided by the total pressure between the parts. Construct a friction curve with v as abscissa.

What information about speed and friction does the curve reveal?

5. It costs \$6 to make an article. The proprietor finds that he can sell in a given time the following numbers of the article at the corresponding prices:

Selling price = S =	\$6	\$12	\$18	\$24	\$30
Number sold = N =	38,000	35,000	24,000	8000	6000

(*a*) Construct a sales curve using S as abscissa.

(*b*) Calculate the profit at each price, $P = N(S - 6)$ and construct curve with same abscissas as in (*a*) with P as ordinates.

(*c*) From the sales curve estimate the number that could be sold at \$15 and at \$20. Insert the values of P corresponding to these in the profit curve. Now draw the profit curve so as to pass through these points.

(*d*) From the profit curve determine the selling price that will give the greatest total profit.

(*e*) What business condition is indicated by the drop in the sales curve from \$12 to \$24? What inference can you draw from the nearly equal sales at \$24 and \$30? From those at \$12 and \$6?

6. The magnifying power (in linear dimensions) of a certain spyglass at different distances from the object viewed was measured as follows:

Magnifying power = m =	9	8.3	8.1	7.6	7.5	7.6	7.6
Distance, meters = x =	2	3	4	5	6	7	8

Construct the magnification curve, using x as abscissa. What general information does the curve reveal? What do you suspect regarding the correctness of the measurement 7.5? The curve should not be made to pass through this value as it is obviously wrong.

7. The sun's declination as given in the nautical almanac for 1915, near the time of the vernal equinox, is as follows:

March 18 noon		South	1° 05′
" 19	"	"	0° 41′
" 20	"	"	0° 18′
" 21	"	"	0° 06′
" 22	"	North	0° 30′
" 23	"	"	0° 51′

Construct the declination curve, using time as abscissa. Take a scale of $\frac{1}{16}''$ to 1 hr. This will enable the determination of hours with fair certainty. Determine the day and hour at which the curve crosses the axis of abscissas. This is the time of the vernal equinox. What is the time at which the sun is exactly over the equator?

Note. — The declination of a heavenly body is its distance north or south of the equator of the sky just as latitude is the distance of a place on the earth north or south of the equator. Declination and latitude are both measured in angular units, that is, in degrees.

8. If s is the amount (grams) of potassium bromide that will dissolve in 100 grams of water at $t°$ centigrade, construct a solubility curve from the following data, using t as abscissa.

$s =$	53.4	64.6	74.6	87.7	93.5
$t =$	0	20	40	60	80

From the curve determine the probable amount that will dissolve at 10°, 30°, 50°, 70°.

9. With the same notation as in (8) the following data were taken for sodium nitrate:

$s =$	68.8	72.9	87.5	102
$t =$	-6	0	20	40

Construct the solubility curve and determine the probable amount that will dissolve at 10°, 30°.

10. Construct the temperature curve from the maximum daily temperatures at a certain city for the month of February. For convenience assume the maximum occurred at the same hour each day. Use time as abscissa.

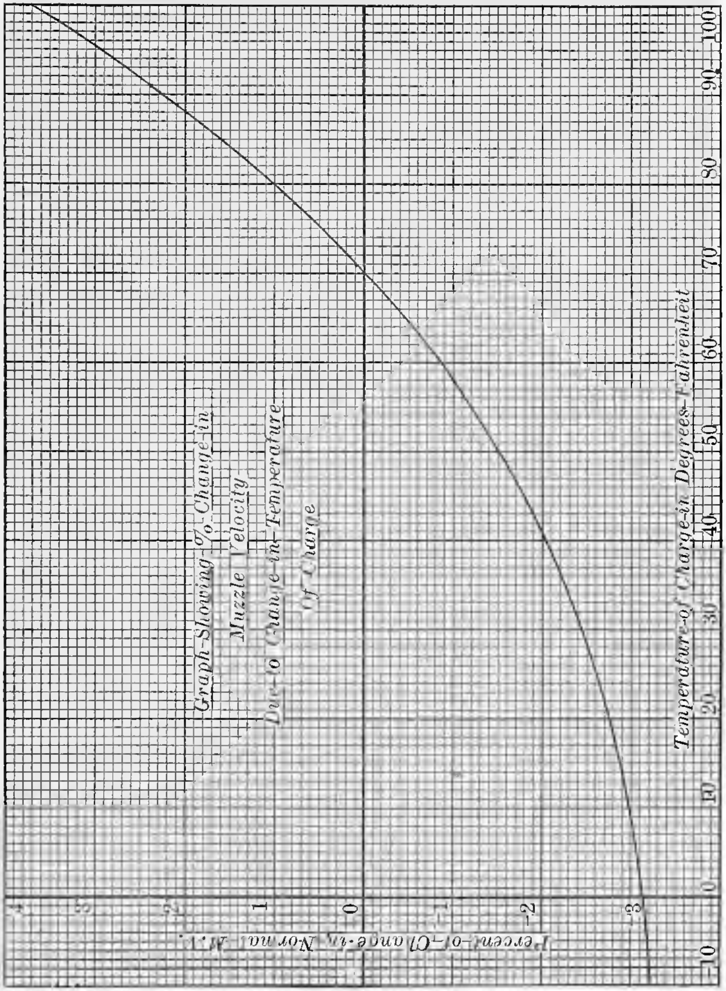
Day.	Temp.	Day.	Temp.	Day.	Temp.	Day.	Temp.
1	28	8	−2	15	10	22	26
2	36	9	14	16	13	23	13
3	31	10	4	17	21	24	20
4	14	11	1	18	24	25	26
5	24	12	11	19	11	26	35
6	28	13	15	20	20	27	42
7	18	14	14	21	30	28	42

11. A 300-lb. projectile is shot from a 10″ rifle with a 70-lb. charge of powder. The speed and powder pressure at different distances from the starting point were measured as follows:

Distance, ft.	Measured speed, ft./sec.	Theoretical speed, ft./sec.	Pressure, lbs./sq. in.
0.41	436.0	450.8	34,752
0.50	501.7	505.0	34,577
0.82	657.0	661.0	32,000
1.06	750.0	748.2	29,648
1.65	911.0	908.0	24,400
2.06	992.8	992.5	21,403
2.26	1027.0	1027.0	19,302
2.74	1097.0	1098.0	17,580
3.06	1137.0	1139.0	16,121
7.06	1412.0	1420.0	7,022
8.26	1464.0	1464.0	5,476
10.15	1527.0	1516.0	4,269

Construct the speed curve and the pressure curve, using distance as abscissa in both cases. In fact both curves may be made on the same axes with different colored pencils. What information regarding speed and pressure at different points in the gun do these curves reveal?

12. Experiments with projectiles show that there is a variation of muzzle velocity due to a variation of the temperature of the powder. The curve on the following page represents the functional relation of temperature of the powder and per cent change in muzzle velocity. For temperature above 70° the arithmetical correction of the velocity is added, for temperature below 70° it is subtracted.


Graph Showing % Change in
Muzzle Velocity
Due to Change in Temperature
Of Charge
Temperature of Charge in Degrees Fahrenheit
Percent of Change in Normal M.V.
-10
0
10
20
30
40
50
60
70
80
90
100
4
3
2
1
0
-1
-2
-3

1. The muzzle velocity of a certain gun is 980 ft./sec. when the temperature of the powder is 70°. Show that 20 ft. should be subtracted from this velocity if the temperature of the powder is 41°.

2. The muzzle velocity of a gun is 2250 ft./sec. when the temperature of the powder is 70°. Find what correction should be made to this velocity when the temperature of the powder is 71.4°.

3. The muzzle velocity of a gun is 2000 ft./sec. when the temperature of the powder is 70°. Find what corrections should be made to this velocity when the temperature of the powder is 33.6°.

CHAPTER V

RATIO, PROPORTION AND VARIATION

29. Whenever four numbers a, b, c, d satisfy the equation

$$a/b = c/d, \tag{1}$$

these numbers are said to form a **proportion** or to be in proportion. The equation (1) contains two equal fractions or ratios.

When we measure any quantity by comparing it with another of the same kind we are said to measure the first quantity by the second as a unit. The number which expresses how many units and parts of units result from the comparison is the measure of the quantity to the unit employed.

30. A **ratio** is defined as the measure of any quantity, abstract or concrete, when some other quantity of the same kind is the unit of measure. When the length of a room is measured with a foot rule (one-foot length) and there is obtained 24 as the measure or numerical value of the length of the room to that unit, it is said the length of the room is 24′. This implies

$$\frac{\text{length of room}}{\text{one foot length}} = 24. \tag{2}$$

This is the idea involved in all measurements where definite units are available.

If the length of the room is measured with a yard stick (one yard length), the measure of the room is 8. The room is said to be 8 yds. long. This implies

$$\frac{\text{length of room}}{\text{one yd. length}} = 8. \tag{3}$$

A ratio is to be regarded as a mere number without material denomination, that is, it is an abstract number. Since a ratio is essentially a quotient or an indicated division, it is con-

veniently represented by a fraction. A ratio is, therefore, subject to all the arithmetical operations ordinarily performed with fractions.

31. From (1) and some general axioms regarding equalities it is easy to establish the following fundamental and useful theorems:

If $\frac{a}{b} = \frac{c}{d}$, then it can be proved that:

1. $ad = bc$, the product of the extremes equals the product of the means.

2. $\frac{a}{c} = \frac{b}{d}$, proportion by alternation.

3. $\frac{b}{a} = \frac{d}{c}$, proportion by inversion.

4. $\frac{a+b}{b} = \frac{c+d}{d}$, proportion by composition.

5. $\frac{a-b}{b} = \frac{c-d}{d}$, proportion by division.

6. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, proportion by composition and division.

7. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = r$, then $\frac{a+c+e+g}{b+d+f+h} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$.

32. Variation. — The statements, "x is proportional to y," "x varies as y," "x is directly proportional to y," "x varies directly as y," are all different ways of saying the same thing. This relation between x and y may be put in the form

(1) $$x = ky,$$

where k is a fixed value, and x, y take any number of different values which satisfy this equation.

The statements, "x varies inversely as y," "x is inversely proportional to y," are each equivalent to the equation

(2) $$xy = k \text{ or } x = \frac{k}{y},$$

where k is constant as in equation (1).

The statements, "x is jointly proportional to y and z," "x varies jointly as y and z," are each equivalent to the equation

(3) $$x = kyz.$$

From type forms (1), (2) there follows a single equation

(4) $$x = \frac{ky}{z}.$$

Equations (1), (2), (3), (4) are to be regarded as special and convenient working forms of proportion. Many laws of nature are expressed in one or another of these forms.

Illustrations. — $f = k\dfrac{m \cdot m}{d^2}$, the law of gravitation; $A = xy$, the area of a rectangle; $E = kn^2$, the law of mob energy.

1. Suppose it is known that s varies as t, and that $s = 25$, when $t = 5$. To find the equation of relation between s and t and to find the value of s when $t = 10$.

Write by (1),

$$s = kt.$$

Substituting values given in the problem,

$$25 = k \cdot 5,$$

therefore $$k = 5.$$

The desired equation is

$$s = 5\,t.$$

The value of s when $t = 10$ is

$$s = 5 \cdot 10 = 50.$$

2. The carrying capacity of pipes varies as the squares of their diameters (friction neglected). How many 6″ pipes will carry as much as one 24″ pipe?

Call the carrying capacity of a pipe C, and write

(1) $$C = kd^2$$

for any and all pipes. For 6″ pipes then,

(2) $$C_6 = k \cdot 6^2 = 36\,k.$$

(3) Also $$C_{24} = k \cdot 24^2 = 576\,k.$$

By equations (2) and (3),

(4) $$\frac{C_{24}}{C_6} = \frac{576}{36} = 16, \text{ the required number of } 6'' \text{ pipes.}$$

Note. — In this case we did not determine the value of k. Its presence in the equations gave us all the advantage of knowing its value without finding it.

3. The weight of a sphere of given material varies as the cube of its radius. If a sphere of $1'$ radius weighs 600 lbs., what will a sphere of radius $5\frac{1}{3}'$ of same material weigh?

Note. — Solve this and the following by the methods of Ex. (1), (2), above.

4. The capacity of a cylindrical tank varies as its height when the diameter is fixed and as the square of the diameter when the height is fixed. A tank $1'$ in diameter and $1'$ high has a capacity of 5.83 gallons. Find the capacity of a tank $8'$ in diameter and $30'$ high.

Note. — Write $C = k \cdot d^2 \cdot h$.

5. The value of a diamond varies as the square of its weight. A diamond worth \$600 is cut in two pieces whose weights are as 1 to 3. What is the value of each piece?

6. The illumination from a light varies inversely as the square of the distance from the light. If an object $10''$ from the light be moved $10\,(\sqrt{5} - 1)''$ farther away, what is the ratio of the final to the original illumination. (Assume I as the original illumination.)

7. The cross section of a chimney should vary as the quantity of fuel used per hr. and inversely as the square root of the height. The cross section of a chimney $150'$ high is 30 sq. ft., the quantity of fuel used per hr. is 15,000 lbs. Find the cross section of a chimney $100'$ high connected to a furnace using 5000 lbs. of fuel per hr.

8. A solid spherical mass of clay $4''$ in diameter is moulded into a spherical shell whose outside diameter is $6''$. What is

the inside diameter of the shell? It is given that the volume of a sphere varies as the cube of its diameter.

9. A safe load on a horizontal beam supported at its ends varies as the breadth and as the square of the depth and inversely as the length of the beam. A beam 2″ x 6″ x 12′, on edge, will sustain a load of 700 lbs. What load will a beam of the same material 3″ x 9″ x 18′, on edge, sustain?

10. The weight of a body above the earth's surface varies inversely as the square of its distance from the center of the earth. If a body weighs 150 lbs. just outside the surface, how high must it be raised so its weight will be 30 lbs., the radius of the earth being assumed to be 4000 mi?

11. A tree casts a shadow 70′ long on level ground. At the same time a 10′ pole casts a shadow 9′ long. Find the height of the tree.

12. The velocity of a falling body starting from rest varies as the time. At the end of 2 sec. the velocity is 64.32′ per sec. What is the formula holding between velocity and time and what is the velocity at the end of 4 seconds?

Note. — Assume $v = k \cdot t$.

13. The interest on a given principle varies jointly as the time and rate. $500 yields $25 in two years. What is the rate?

14. If the amount of fuel required to heat a house varies as the square of the difference in temperature in the house and outside and if when the thermometer reads 0° C outside, and the temperature inside is 20° C, it requires 1000 cu. ft. of gas per hr. to heat a certain house, how much gas would be necessary to heat the house to the same temperature when the thermometer outside dropped to −10° C?

SUPPLEMENTARY EXERCISES

1. If y varies as x and if $x = 4$ when $y = \frac{3}{2}$, find y when $x = 3$.

2. If y is proportional to x and if $x = \frac{3}{4}$ when $y = \frac{2}{3}$, find x when $y = 7$.

3. If y varies directly as x and inversely as z and if $y = 8$ when $x = 12$ and $z = \frac{4}{3}$, find z when $y = 2$ and $x = 6$.

4. If y varies jointly as x and z and if $y = \frac{2}{3}$ when $x = 5$ and $z = \frac{3}{5}$, find z when $y = 3$ and $x = 6$.

5. If y varies as x and $x = 6$ when $y = 54$, what is the value of y when $x = 8$?

6. If x varies directly as y and inversely as z, and $x = \frac{2}{3}$ when $y = 20$ and $z = 10$, what is the value of x when $y = 9$ and $z = 20$?

7. If x^4 is directly proportional to y^3 and $x = 4$ when $y = 4$, what is the value of x when $y = 9$?

8. If $2x - 3$ is proportional to $y + 6$ and $x = 4$ when $y = 3$, what is the value of y when $x = 5$?

9. The hypotenuse of a right triangle is 100 ft. long. Find the other sides, if their ratio is 3 to 4.

10. The stretch in the spring of an ordinary spring balance is proportional to the weight (force) applied. If a force of 4 lbs. stretches it one inch, how much will a force of 17 lbs. stretch it?

11. How would you graduate (divide) a scale for the spring balance in the above problem?

12. The area of a circle varies as the square of its diameter. How is the diameter affected when the area is doubled? How is the area affected when the diameter is doubled? Give work showing reasons for your answers.

13. If two pulleys are connected with a belt prove:

(a) That the number of revolutions that they make per minute are to each other inversely as their diameters.

(b) That their radii are to each other inversely as their speeds.

14. If x varies as y, then $x = ky$ and conversely, if $x = ky$, then x varies as y. Show that the speed of a point on the rim of a pulley varies as its diameter.

15. If in problem 13 the driving pulley is making 20 revolutions per minute, and its diameter is 10 inches, find the number of revolutions per minute of the second pulley if it is 4 inches in diameter.

16. According to Boyle's law of gases, pressure (p) times volume (v) is constant. How does the pressure vary with the volume? Show graphically the relation between (p) and (v) if $v = 1$ cu. ft. when $p = 25$ lbs. per sq. in.

17. Given that:

$$L = 2\pi rh \qquad \text{and } L' = 2\pi r'h',$$
$$A = 2\pi r(r + h) \qquad \text{and } A' = 2\pi r'(r' + h'),$$

formulas which represent the total area and lateral area of two right circular cylinders. Show that

$$\frac{L}{L'} = \frac{A}{A'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

18. In reading contour maps the question arises whether a station B is

visible from some station A. The problem is to determine whether an intervening height of ground C obstructs the line of sight from A to B.

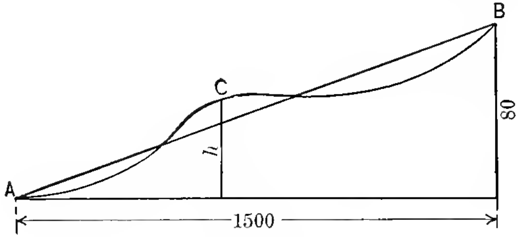


FIG. 10.

The horizontal distance between A and B as scaled on the map is 1500 yds., between A and C is 900 yds., and height of B above datum plane (horizontal plane through station A) is 80 ft., height of obstacle C above datum plane is 50 ft. Determine whether A is visible from B. By means of the figure and theorem 27, Chapter II, it is seen that:

$$900 : 1500 = h : 80$$

and solving

$$h = 48.$$

Therefore, the line joining stations A and B would pass station C at a height of 48 ft. above the datum plane. Since obstacle C is 50 ft. above this plane it obstructs the vision from B to A.

19. Check the above problem by drawing the figure to scale.

20. In problem 18 substitute y for 900, x for 1500, H for 80, and solve the equation for h. Draw a figure and indicate the distances x, y, H and h.

21. Using the result of problem 20, find whether C would obstruct the line of sight from B to A if $x = 2100$ yds., $y = 1400$ yds., $H = 130$ ft. and the height of C above datum plane is 95 ft.

CHAPTER VI

THE RECTANGULAR COÖRDINATE SYSTEM: GRAPHS OF EQUATIONS; FORMULAS

33. Two lines are selected, intersecting at right angles, in the plane of the paper (they are usually chosen the one horizontal and the other vertical). The **position** of a **point** is known if its distances from these two lines are known. Let

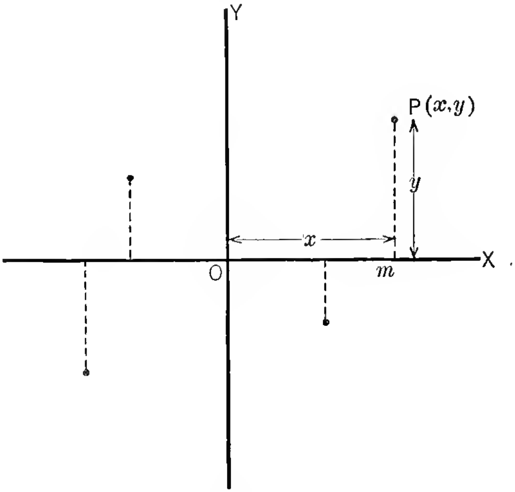


FIG. 11.

XX' and YY' intersect at right angles in O. The position of P is known if $Om = x$ and $mP = y$ are known. The point P will often be designated by $P\ (x, y)$ or by (x, y), at pleasure. It is noted that the abscissa x is always written before the ordinate y and this arrangement holds no matter what letters are used to represent the coördinates.

Since the point P may lie either to the right or left of YY' and above or below XX', we must have further conventions. The following have been universally adopted:

The ordinates of points above the XX' axis are positive.

The ordinates of points below the XX' axis are negative.

The abscissas of points to the right of YY' axis are positive.

The abscissas of points to the left of YY' axis are negative.

The point O is called the **origin** of coördinates or, for short, the origin. Both its coördinates are zero.

1. Locate the following points in rectangular coördinates: $(2, -5)$; $(-4, 1)$; $(-3, -3)$; $(4, 5)$; $(1, 8)$.

2. What is the abscissa of any point on YY'? What is the ordinate of any point on XX'?

3. What is the ordinate of any point of a line parallel to XX' and 6 units above it? On what line do all points whose abscissas equal -3 lie?

34. Graphs of functions * and equations. — To construct the graph of any function, say $3x^2 - 4x + 5$, write it equal to some symbol, say y, obtaining an equation,

$$y = 3x^2 - 4x + 5. \tag{1}$$

Select one of the symbols x and y for the abscissa and the other as the ordinate of points on the graph. In this and similar cases, x is usually the abscissa and y the ordinate.

Every equation like (1) expresses a relation between different **number pairs.** Every number pair are the coördinates of a point. One number of each pair may be chosen at pleasure, the other number of the pair must be calculated from the equation. Any number of such number pairs may thus be determined. The corresponding points can be located on the diagram. Having located several points, draw a smooth curve through these points as in Chapter IV. This curve is called the graph or the locus of the equation.

It must be remembered that *the coördinates of all points on*

* For definition of function see Chapter VII. The term formula might be used at present.

the graph must satisfy the equation. On the other hand *every number pair which satisfy the equation must be coördinates of a point on the graph.* To be sure, the absolutely exact graph cannot be drawn without determining the coördinates of every point on it. This, obviously, would be impossible for it would involve endless calculation to obtain even a small portion of the graph. But by carefully determining a few well-selected points of the graph and then drawing a smooth curve through these points, it will be found that a very close approximation to the true graph is obtained. This approximate graph is sufficiently accurate to permit interpolation and to furnish a good basis for a study of the equation it represents and of the true graph.

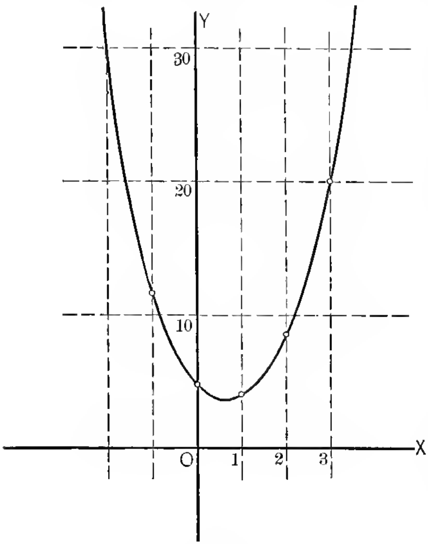


FIG. 12.

The process of **constructing the graph** of equation (1) is indicated below. Choose values of x and calculate the corresponding values of y from the equation,

$$x = \quad -2, \quad -1, \quad 0, \quad +1, \quad +2, \quad +3$$
$$y = +25, \quad +12, \quad +5, \quad +4, \quad +9, \quad +20$$

The annexed figure shows the approximate graph which gives a good idea of the true graph. Owing to the large values of y, a smaller scale is used for ordinates than for abscissas in order to reduce the space necessary to draw a sufficient portion of the curve. This distortion modifies the form of the curve but does not affect its fundamental nature. It reduces the distance between the calculated points of the curve, see **17.**

2. Construct the graph of the equation $2x + 3y = 5$. Assign values to x and calculate values of y from the equation.

$x =$	-3	2	5	8
$y =$	$3\frac{2}{3}$	$\frac{1}{3}$	$-1\frac{2}{3}$	$-3\frac{2}{3}$

This locus is a straight line. What is its slope? At what value of x does the line cross the axis XX'? This value is the x-intercept of the line. At what value of y does the line cross the axis YY'? This value is the y-intercept of the line.

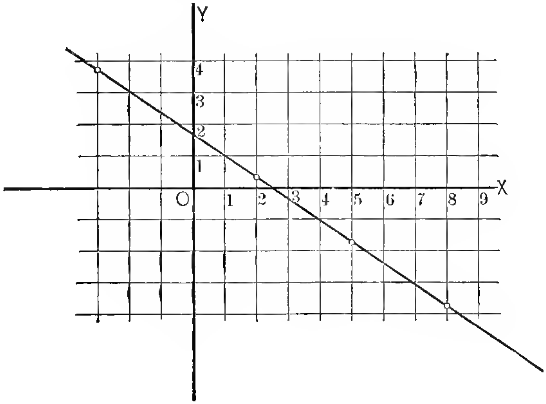


FIG. 13.

A straight line is always associated with an equation of the first degree in two unknowns.

3. Construct the graph of the equation $x^2 + y^2 = 25$. As in the preceding cases choose values of x and calculate the corresponding values of y from the equation.

$x =$	-6	-5	-4	-3	-2	-1	0
$y =$	(i)*	0	± 3	± 4	± 4.6	± 4.9	± 5
$x =$	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$	
$y =$	± 4.9	± 4.6	± 4	± 3	0	(i)*	

It is noted that to every value of x there correspond two equal but oppositely signed values of y. The curve is, therefore,

* (i) represents an imaginary quantity.

symmetrical about the axis XX'. If values of y are chosen and values of x calculated from the equation it will be found that to every value assigned to y there correspond two equal and opposite values of x. Therefore the curve is symmetrical about the axis YY'. Consequently the curve is symmetrical about the origin. This curve is a circle. Its radius is 5 and its center is at O. A circle is always associated with an equation of the form $x^2 + y^2 = a^2$, that is, the sum of two squares equals a constant.

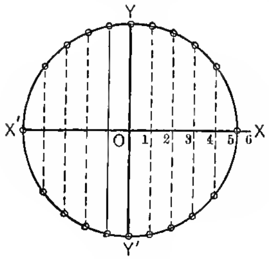


FIG. 14.

What is the nature of the values of			y for all values of		$x < -5$?
"	"	"	y	"	$x > 5$?
"	"	"	x	"	$y < -5$?
"	"	"	x	"	$y > 5$?

It is seen that no real values of x or y outside the limits -5 and $+5$ will satisfy the equation. This means that no points of the curve can lie outside these limits.

4. The corresponding values of two symbols x and y are such that the square of one equals the square of the other increased by 1. Write the equation and construct the graph. Save graph for Ex. 7.

5. The corresponding values of two symbols are such that the sum of their squares equals 49. Write the equation and construct the graph. Save graph for Ex. 7.

6. Construct the graphs of $y = 2\,x$ and of $y = 2\,x + 6$ on the same axes and to same scale. What likeness and what difference do you notice regarding the lines? What likeness and what difference regarding the equations?

What is the slope of each line? What is the coefficient of x in each equation? Can you tell the slope of a straight line from its equation without drawing the line? How?

7. Determine intercepts on the axes of all the lines in the last three exercises.

8. Construct the graph of $x + 4y = 14$ and determine the slope and the intercept on the axes.

9. Construct the graphs of $x/3 + y/5 = 9$ and $8x - 4y + 4 = 0$ on the same diagram. Determine by measurement the coördinates of the point of intersection of these graphs. Solve the given equations as simultaneous equations for x and y. Compare the results with the coördinates of the point of intersection. Explain.

10. Treat the equations $x^2 + y^2 = 25$ and $x - y = 1$, in a manner similar to that directed in Ex. 9.

11. Treat the equations $x + y = 6$ and $x - y = 1$ in a manner similar to that directed in Ex. 9.

12. Draw the triangle whose sides lie in the lines represented by the equations: $x + y = 4$; $2x + y = -2$; $x - y = -6$.

Note. — The vertices of the triangle will be the three points of intersection of the lines in pairs.

13. A sphere of wood 1′ in diameter sinks in water to a depth determined by one root of the equation $2x^3 - 3x^2 + 0.657 = 0$.

Note. — The desired root will be one of the x-intercepts of the curve, $y = 2x^3 - 3x^2 + 0.657$. Explain.

35. An **important use** of graphs is found in the determination of **empirical formulas** expressing laws of nature. From observations made in the laboratory or in the field several number pairs are measured. From these a curve is constructed. This curve represents to the eye the relation between the numbers of each of the number pairs. The form of the curve may often suggest to the experienced mathematician the general form of an equation of which the curve is the graph. It remains to determine the constants of this equation. Sometimes only an approximately correct formula can be determined at first. This formula is subject to later correction by additional observations and by the application of least squares.

A few examples will illustrate the method of procedure.

1. Let us attempt to find an equation for problem 6, **28,** Chapter IV. The form of the curve suggests to one experienced in the art, an equation of the form $xy = k$ or $y =$

$\frac{a}{x} + b$ as possibilities where either k or else a and b are to be determined. Let us try the second form and write

$$m = \frac{a}{x} + b. \tag{1}$$

There being two unknown constants we shall need two equations. These are obtained by substituting in (1) two pairs of observed values of m and x. Thus

$$9 = \frac{a}{2} + b, \tag{2}$$

$$7.6 = \frac{a}{5} + b. \tag{3}$$

Solving these equations simultaneously for a and b gives $a = 4.7$ and $b = 6.7$. Substituting these in equation (1) gives as the desired formula:

$$m = \frac{4.7}{x} + 6.7. \tag{4}$$

To check the validity of this formula for values not included in determining a and b, put $x = 4$ and m turns out to be 7.9. The corresponding observed value is 8.1. The formula gives fairly good results considering the nature of the quantities concerned.

2. An innkeeper finds that if he has G guests per day his expenses are $\$E$ and his receipts $\$R$. His books furnish the following data:

$G =$	210	270	320	360
$E =$	83	97	108	117
$R =$	79	106	132	149

Using G as abscissa construct two graphs, one for E and one for R on same axes and same scale. These lines appear to be straight lines. This fact suggests an equation for each of the form $y = mx + b$. Write

(1) $$E = m_1G + b_1.$$
(2) $$R = m_2G + b_2.$$

The values of m_1, m_2, b_1, b_2 may be determined easily from the graph. The slopes are m_1, m_2. Measure $m_1 = \frac{dc}{db} = 0.203$; $m_2 = \frac{d'c'}{d'b'} = 0.444$. The y-intercepts are $ob = b_1 = 32$; $ob' = b_2 = -35$. These values are to be regarded as close approxi-

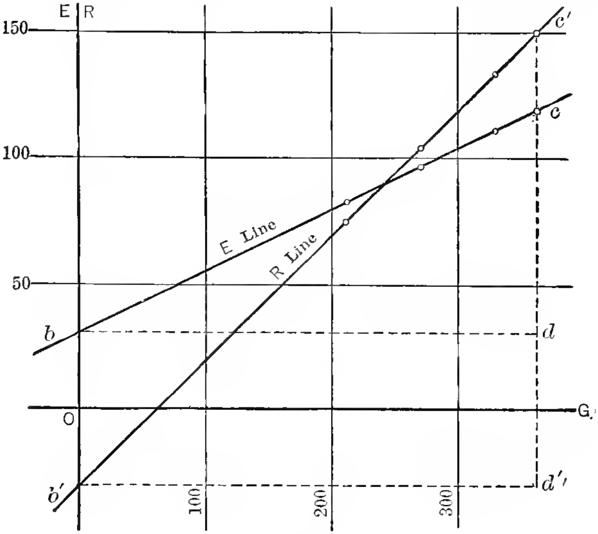


FIG. 15.

mations, only. The equations (1) and (2) now become, by substituting these values,

(1′) $$E = 0.203\,G + 32.$$
(2′) $$R = 0.444\,G - 35.$$

(*a*) What number of guests is just sufficient to pay expenses?

(*b*) What are the expenses when G is zero? What are the profits when $G = 360$? How can profits and losses be determined from the graph?

Equations (1′) and (2′) can be obtained algebraically. Substitute two pairs of values of E and G in (1) and solve for m_1, b_1. Then substitute two pairs of values of R and G in (2) and solve for m_2 and b_2. Thus from (1)

$$83 = m_1 \cdot 0.210 + b_1,$$
$$117 = m_1 \cdot 360 + b_1.$$

Whence $m_1 = 0.233$ and $b_1 = 35$. In a similar manner with Eq. (2):

$$79 = m_2 \cdot 210 + b_2,$$
$$149 = m_2 \cdot 360 + b_2.$$

Whenee $m_2 = 0.466$ and $b_2 = 29$. These values are in fair agreement with the ones determined above.

3. The latent heat of steam at $\theta°$ C. is L. Construet a curve from the values given below and determine an equation of the form $L = m\theta + b$.

4. The height h, above the earth's surface and the corresponding barometer reading p, are as follows:

h (ft.) =	0	886	2703	4763	6942
p (in.) =	30	29	27	25	23

Construct the graph and determine an equation of the form

$$p = Ae^{kh}.$$

Note. — Take the logarithm of both sides of the proposed equation and proceed as in previous cases. Thus

$$\log p = \log A + kh \log e, \qquad e = 2.718.$$

Determine first, log A and k as in previous cases. The value of A and k are then to be substituted in the proposed equation $p = Ae^{kh}$.

5. Determine the equation of the form $y = ax^2 + bx + c$ of the curve which passes through the points given by

$x =$	2	3	4	5
$y =$	10	-6	-5	10

6. Construct the curve and determine the equation of the form $y = x^3 + bx^2 + cx + d$ from the following data:

$x = -2$	-1	0	1	2	3
$y = -8$	-1	0	1	2	27

7. Construct the graph and determine an equation of the form $y^2 = ax + b$ from

$x = 2$	4
$y = 4$	10

Determine whether or not the points (0, 0), (3, 7) are off the curve.

8. Determine an equation and draw the curve from

$x = 0$	1	3	6
$y = 6$	5.9	5.3	0

The equation is to be of the form $ax^2 + by^2 = c^2$.

9. For an ideal gas Boyle's law says that the product of the pressure and volume of a given mass of gas at constant temperature is constant. Form an equation from this statement and draw the graph. Determine an equation from the following:

Pressure, inches of mercury =	130	45	60	75	90	105
Volume, cubic centimeter =	100	66.6	50	40	33.3	28.5

10. The intensity of illumination from a light varies inversely as the square of the distance from the light. Write this in the form of an equation and draw the graph. Determine an equation from

$L = 100$	50	25	5
$D = 10$	14.14	20	44

Note.—If desired the subject of empirical formulas may be continued at this time by taking up the work, which appears in a later chapter, on the applications of logarithmic and semi-logarithmic paper in determining certain types of empirical formulas.

CHAPTER VII

NUMBERS, VARIABLES, FUNCTIONS, LIMITS

36. Numbers. — (*a*) The numbers 1, 2, 3, . . . used in ordinary counting are called the natural or absolute integers. The idea of positive and negative does not belong to them. They answer the question, "How many?" We associate with the natural integers all fractions formed with them, such as, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{17}{9}$, ... The natural integers and the associated fractions constitute the number system of ordinary arithmetic. They are sometimes called non-directed numbers. They are sometimes, but with questionable propriety, identified with the positive numbers of algebra.

(*b*) The idea of **positive** and **negative** numbers is a relatively modern notion. Certain natural phenomena, scientific measurements and the fact that subtraction was not always possible with the natural numbers suggested the idea of "opposite" numbers or positive and negative numbers.

(*c*) A **rational number** is one that can be expressed as the quotient of two integers, positive or negative. Some rational numbers are expressible in decimal form as $\frac{3}{1}$, $\frac{2}{5} = 0.4$, $-\frac{1}{8} = -0.125$, etc. Some rational numbers cannot be exactly expressed in decimal form, as $\frac{2}{3} = 0.666\ .\ .\ .$, $\frac{1}{9} = 0.1111\ .\ .\ .\ .$

(*d*) **Irrational numbers** result from certain operations on rational numbers. Thus $\sqrt{2} = 1.4142\ .\ .\ .$ is irrational. For it cannot be expressed as the quotient of two integers. The numbers $\sqrt{3}$, $\sqrt[3]{\frac{1}{7}}$, $\pi = 3.14159\ .\ .\ .$ are examples of irrational numbers. The logarithms of most numbers are irrational.

(*e*) All the numbers so far mentioned come under a more general class called *real numbers*. The name is derived from their association with ordinary affairs and by contrast with a class of numbers defined below.

(*f*) An **imaginary number** is one that arises in the attempt to take an even root of a negative real number. Thus $\sqrt{-2}$, $\sqrt[4]{-3}$, . . . are imaginary numbers.

Remark. — It should be noted that the name "imaginary" applied to numbers reflects an attitude of mind, existing formerly, toward such numbers. Recent developments have shown that term is unfortunate. For imaginary numbers have come to have a very real meaning in scientific investigations. Attention seems first to have been directed to imaginary numbers in the study of quadratic equations.

(*g*) The sum of a real number and an imaginary number is called a **complex number**. All complex numbers are of the form $a \pm b\sqrt{-1}$ where a, b are real. Thus $2 + \sqrt{-9} = 2 + 3\sqrt{-1}$; $-\frac{1}{2} + \frac{1}{2}\sqrt{-3} = -\frac{1}{2} + \frac{1}{2}\sqrt{3} \cdot \sqrt{-1}$ are complex numbers. Use will be made of complex numbers in a later chapter.

It is to be kept in mind that in practical calculations involving irrational numbers it is necessary to use a rational number nearly equal to the corresponding irrational number. Thus for $\sqrt{2}$ the rational number 1.4142, which is correct to five figures, may be used, and similarly in other cases.

37. (*a*) A **variable** is a symbol to which, in a given problem, may be assigned an indefinitely great number of values, and which may be employed in calculations as a number. The values assigned to a variable may be assigned in accordance with some law of nature or in accordance with some arbitrary mode of thought.

May x and y have more than one value each in the equation $y - 2x = 4$? Does 2 ever have any value different from 2? Are x and y variables or fixed in the equation? Is 2 variable or fixed? Can you define a constant?

(*b*) When a variable assumes or may assume every assignable value between two given constant values the variable is said to be **continuous** or to vary in a continuous manner, in the interval between the given constants.

The totality of values between two given values is called an **interval.** If the given values are included the interval is closed, if not the interval is open.

If a variable may not assume all values in a given interval it is **not continuous** in that interval, though it may be continuous in parts of the interval.

Suppose a bottle of ink stands, uncorked, on the table for several days or weeks, and suppose it is not disturbed in any way. Is the amount of ink in the bottle from time to time the same? Is the quantity of ink constant? Does it vary continuously?

Suppose wheat is \$1.75 per bu. today, \$1.90 yesterday and \$2.00 tomorrow. Is the price of wheat constant? Does it vary continuously?

(*c*) Any set of numbers taken in some definite order is called a *sequence.* If there are infinitely many numbers in the set the sequence is called an **infinite sequence.** If there are only a finite number of numbers in the set it is a **finite sequence.** Thus

$$1, \quad 3, \quad 4, \quad 7, \quad 9, \quad 15,$$

form a finite sequence. But

$$1, \quad 1.1, \quad 1.11, \quad 1.111, \ldots$$

form an infinite sequence. A sequence may increase or decrease. That is, the successive numbers may be larger than or smaller than the preceding, respectively.

(*d*) Let x be a variable and a some constant. Then if $a - x$ assumes its sequence of values in order, there comes a stage, such that, for all subsequent values of x, the numerical value of $a - x$ becomes and remains less than any assigned small value ϵ, then a is called the **limit** of x.

This definition is expressed in symbols as $a =$ limit x, or $x \rightarrow a$ or $x \doteq a$. All these forms may be read a is the limit of x or x approaches a as a limit.

A variable **may** or **may not** become equal to its limit, depending on the nature of the law of its change.

The difference between a variable and its limit is a variable whose limit is zero.

A variable whose limit is zero is called an **infinitesimal.**

A variable that may increase without limit is said to have no limit or with less propriety to have **infinity** for its limit.

(*e*) Let y be a variable and x another variable. If to every value assigned to x there corresponds a definite value assumed by y, then y is a **function** of x.

More concretely but less exactly it may be said that the values of y, the function, **depend** upon the values of x, the variable. From this notion we have the terms **dependent variable** or function and **independent variable** or merely variable. In the above definition x is the independent variable and y the dependent variable. In dealing with equations and their graphs it is customary to regard the abscissa as the independent variable and the ordinate, the dependent variable or function.

38. When it is desired to express briefly the fact that y is a function of x we may write:

$y = f(x)$ (read y equals f of x or f function of x) or $y = \phi(x)$, etc.

This is a notation used for convenience. Suppose y is defined as a function of x by the expression,

$$y = f(x) = 4\,x^3 - 6\,x + 4.^*$$

This equation defines $f(x)$ for this particular case and during the discussion of this function $f(x)$ is understood as a brief way of writing the polynomial $4\,x^3 - 6\,x + 4$. When particular values are substituted for x the fact is indicated by

$$\begin{aligned} f(3) &= 4 \cdot 3^3 - 6.3 + 4, \\ f(-1) &= 4 \cdot (-1)^3 - 6 \cdot (-1) + 4, \\ f(a) &= 4\,a^3 - 6\,a + 4, \text{ etc.} \end{aligned}$$

If $f(x)$ is any function of x and if $f(a) - f(x) \rightarrow 0$, when $x \rightarrow a$, $f(x)$ is called a continuous function of x at the value $x = a$. It is assumed that $x \rightarrow a$, either by decreasing or by increasing values. This is expressed by writing $|f(a) - f(x)| \rightarrow 0$

* This equation is called a functional equation between x and y.

where the "| |" indicate the absolute value; that is, the numerical value without a plus or minus sign.

39. It is one of the **chief problems** of mathematics to discover and to study functions of variables. This study is most easily carried on when the function is expressed as an equation between the function and the independent variable. Such an equation is called a functional equation. By studying the equation the mathematician learns the character or properties of the function which the equation expresses.

When any natural phenomenon becomes so well known that it can be described by a functional equation, it becomes a part of mathematics. The mathematician may discover further facts and peculiarities of the phenomenon. In this way science and its applications have been greatly extended.

40. The difference between two successive values of a variable is called an **increment** of the variable. Let x_1, x_2 be two successive values of the variable x. Then $x_2 - x_1$ is the increment of x. The symbol Δx * is used to represent the increment of x. Thus $x_2 - x_1 = \Delta x$. When $x_2 > x_1$, $\Delta x > 0$. When $x_2 < x_1$, $\Delta x < 0$. Similar definitions and notations apply to any variable and to functions.

41. Special forms and limits. — Theorems.

I. $\lim\limits_{x \to \infty} \left|\dfrac{a}{x}\right| = 0$, where a is a definite number.

II. $\lim\limits_{x \to \infty} \left|\dfrac{x}{a}\right| = \infty$, where a is a definite number.

III. $\lim\limits_{x \to 0} \left|\dfrac{x}{a}\right| = 0$, where a is a definite number not zero.

IV. $\lim\limits_{x \to 0} \left|\dfrac{a}{x}\right| = \infty$, where a is a definite number not zero.

The student can easily satisfy himself concerning the reasonableness of these theorems by arithmetical methods. For example, consider the sequence of fractions with the same numerators but with increasing denominators,

$$\tfrac{2}{5},\ \tfrac{2}{50},\ \tfrac{2}{500},\ \tfrac{2}{5000},\ \cdots$$

* Read delta x.

From our knowledge of division in arithmetic it is evident that the successive fractions are smaller and smaller in value as we proceed with the sequence. That is, they are nearer and nearer zero. They are approaching zero.

Below are given the usual proofs of the above theorems.

42. I. The limit of $\left|\frac{a}{x}\right|$ as x becomes infinite is zero, that is, $\lim_{x \to \infty} \left|\frac{a}{x}\right| = 0$, where a is a definite number.

Suppose

$$\left|\frac{a}{x}\right| < \epsilon \quad \text{or} \quad \frac{|a|}{|x|} < \epsilon$$

where ϵ is an arbitrarily small number, not 0. Then

$$|a| < \epsilon |x| \quad \text{and} \quad |x| > \frac{|a|}{\epsilon}.$$

Therefore, for any assigned value of ϵ, $\epsilon \neq 0$, we can calculate a value of x such that $\left|\frac{a}{x}\right| < \epsilon$. It follows that for indefinitely increasing values of x, $\left|\frac{a}{x}\right|$ satisfies the definition of limit of a variable and has zero for its limit.

II. $\lim_{x \to \infty} \left|\frac{x}{a}\right| = \infty$, where a is a definite number.

It will be sufficient to show that if x is chosen sufficiently large the inequality $\left|\frac{x}{a}\right| > M$ can be satisfied, however large M is chosen. For multiplying both sides by $|a|$ gives

$$|x| > |a| M.$$

It is only necessary then to choose $x > |a| M$ in order to ensure the inequality $\left|\frac{x}{a}\right| > M$, however large M may be.

III. $\lim_{x \to 0} \left|\frac{x}{a}\right| = 0$, where a is a definite number not zero.

It will be sufficient to show that if x is chosen sufficiently

small the inequality, $\frac{|x|}{|a|} < \epsilon$, however small ϵ be chosen, will be satisfied. Multiplying both sides by $|a|$ gives

$$|x| < \epsilon |a|.$$

Now if a and ϵ are given x is determined at once so as to satisfy the condition $\frac{|x|}{|a|} < \epsilon$.

IV. $\lim_{x \to 0} \left| \frac{a}{x} \right| = \infty$, where a is a definite number not zero.

It will be sufficient to show that if x be chosen sufficiently small the inequality $\frac{|a|}{|x|} > M$, however large M is chosen, holds. Multiplying both sides by $|x|$ gives

$$|a| > M|x|.$$

Dividing now by M,

$$|x| < \frac{|a|}{M}.$$

This value of x will ensure the first inequality.

43. It is often desirable to know the limit approached by an expression when the variable approaches a given value. The preceding theorems are useful for this purpose.

1. What is the limiting value of $\frac{x-2}{x^2 - 4x + 4}$, when $x \to 2$.

By direct substitution of $x = 2$ in the function the result is $0/0$. This result can have no meaning. But it is noticed that this expression is not in its lowest terms. For

$$\frac{x-2}{x^2 - 4x + 4} = \frac{1}{x-2}.$$

Now as $x \doteq 2$, the result is ∞, by IV. The expression $0/0$ may be assigned any value. For, write

$$\frac{x}{y} = k$$

and let $x \to 0$ and $y \to 0$ in such a way that this equation always holds. Multiplying by y gives

$$x = ky.$$

This equation is satisfied even when $x = 0$ and $y = 0$. But k was arbitrary, therefore $\frac{0}{0}$ may have any value and is indeterminate.

2. What is the limit of $\dfrac{x^2 - 6x + 1}{x^3 - 3x^2}$ as $x \to \infty$. By direct substitution the result is ∞/∞ which is equally as indeterminate $0/0$. But if numerator and denominator be divided by x^3 the expression becomes,

$$\frac{\dfrac{1}{x} - \dfrac{6}{x^2} + \dfrac{1}{x^3}}{1 - \dfrac{3}{x}}.$$

Now as $x \to \infty$ all the terms of the numerator become 0, by I. The denominator becomes 1. Therefore the value of the fraction becomes 0, by III.

3. What is the limit of $\dfrac{x-1}{x^3-1}$, when $x \to 1$?

4. What is the limit of $x^2 - 6x + 2$ when $x \to 0$, $x \to 2$, $x \to 10$, respectively?

5. What is the limit of $e^{\frac{1}{x}}$, when $x \to \infty$?

6. What is the limit of $x + \dfrac{1}{x}$, when $x \to \infty$? When $x \to 0$?

7. What is the limit of $\dfrac{x-2}{x^2-1}$, when $x \to 0$?

8. What is the limit of e^{-x}, when $x \to 0$? When $x \to \infty$?

44. It should be noted that in much of the work of previous chapters we were concerned with **number pairs**. We selected one number of a pair and found the other by observation or calculation. This **correspondence** of number pairs satisfies the definition of function. Suppose (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . , (x_n, y_n) be a set of number pairs, such that as a variable x

assumes the values $x_1, x_2, \ldots, x_n$, another variable y assumes the corresponding values, $y_1, y_2, \ldots, y_n$, the variable y is to be regarded as a function of the variable, x, over the given set of values.

If each pair of values of x and y be regarded as the coördinates of a point, a curve drawn through these points represents more or less approximately the function in question. In fact the curve may be regarded as **defining**, approximately, a function, a few of whose values are known, *viz.*, the points used in drawing the curve. **By measuring** the abscissas and ordinates of other points of the curve we may determine any number of other values of the variable and the corresponding values of the function, approximately. It is seen that the graphic representations of previous chapters represent or define, approximately at least, functions, whether we know the functional equations or not. In some cases we were able to discover an equation from the graph.

It is essential to progress in mathematics and its applications that we recognize the use of the graph as a means of studying functions and of discovering the corresponding functional equations, and that we learn to study any function by means of both its equation and its graph.

In the next chapter we shall study a class of functions which have a wide range of application, not only in mathematics but in science and engineering as well. In later chapters we shall study still other kinds of functions.

CHAPTER VIII

THE TRIGONOMETRIC FUNCTIONS

45. Problem. — It is desired to know the height of a tree, BC. It is found that the distance AC and the angle CAB can be measured. From this data it is desired to find the height of the tree, but these measurements alone will not be sufficient

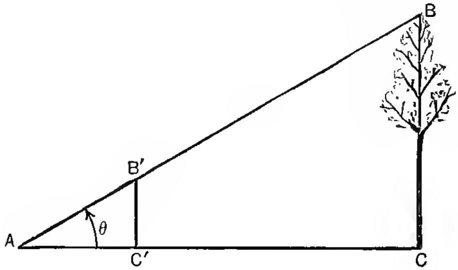


FIG. 20.

for the purpose. The additional measurements AC' $B'C'$ are, therefore, taken. This is done by means of the upright pole $B'C'$ such that $AB'B$, the line of sight, is a straight line. Now from the similar triangles $AC'B'$ and ACB the proportion

(1) $$\frac{B'C'}{AC'} = \frac{BC}{AC},$$

or

(2) $$BC = \frac{B'C'}{AC'} \cdot AC$$

can be written. Equation (2) shows that BC depends on AC and the ratio $B'C'/AC'$. This ratio evidently depends in some way on the angle $CAB = \theta$. The frequent occurrence, in science and engineering, of situations similar to this caused

mathematicians to search for the functional relation of the ratio to the corresponding angle. By careful measurements and calculations the values of the ratio and the corresponding angle have been tabulated for convenience in solving problems.

The ratio $B'C'/AC'$ is called the **tangent ratio** of the angle θ. Having a **table** of such ratios for different angles it is easy to calculate the height of the tree from the original measurements of AC and the angle $\theta = CAB$. Equation (2) may now be written

(3) $$BC = AC \text{ (tangent of } \theta)$$

or more briefly

(4) $$BC = AC \tan \theta.$$

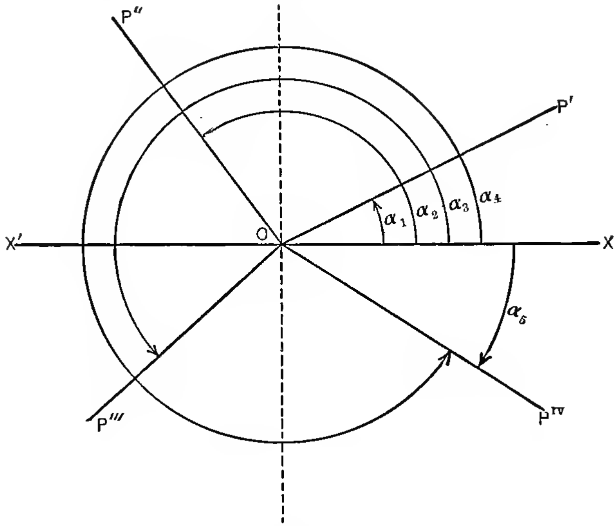


FIG. 21.

In this chapter we shall study the theory and use of the tangent ratio and other related ratios. The methods developed will furnish ways of solving problems of the highest importance in science and engineering.

46. An **angle** will now be regarded as generated by the rotation of a straight line about one of its points from some

initial position to some final or terminal position. The angle will be included between lines lying in these two positions with its vertex at the point of rotation. For the purpose of formulating the definitions and fundamental relations the line OX in Fig. 21 will be taken as the standard initial line or position. Let OP rotate from OX, about O, counterclockwise. Let OP', OP'', OP''', OP^{IV} be successive positions chosen as OP rotates. These lines are the terminal lines of the angles, α_1, α_2, α_3, α_4, respectively, as shown in the figure.

Counterclockwise rotation is regarded as positive and clockwise as negative. This convention gives rise to positive and negative angles to correspond. The angle α_5 is negative.

47. Hessler's definitions of the **trigonometric functions** (ratios) are as follows: (See Fig. 22.)

Name.	Symbol.	α_1 Quad. I.	α_2 Quad. II.	α_3 Quad. III.	α_4 Quad. IV.
sine of α............	$\sin\alpha$	$(+)\frac{y_1}{h_1}$	$(+)\frac{y_2}{h_2}$	$(-)\frac{y_3}{h_3}$	$(-)\frac{y_4}{h_4}$
cosine of α..........	$\cos\alpha$	$(+)\frac{x_1}{h_1}$	$(-)\frac{x_2}{h_2}$	$(-)\frac{x_3}{h_3}$	$(+)\frac{x_4}{h_4}$
tangent of α........	$\tan\alpha$	$(+)\frac{y_1}{x_1}$	$(-)\frac{y_2}{x_2}$	$(+)\frac{y_3}{x_3}$	$(-)\frac{y_4}{x_4}$
cotangent of α......	$\cot\alpha$	$(+)\frac{x_1}{y_1}$	$(-)\frac{x_2}{y_2}$	$(+)\frac{x_3}{y_3}$	$(-)\frac{x_4}{y_4}$
secant of α..........	$\sec\alpha$	$(+)\frac{h_1}{x_1}$	$(-)\frac{h_2}{x_2}$	$(-)\frac{h_3}{x_3}$	$(+)\frac{h_4}{x_4}$
cosecant of α.......	$\csc\alpha$	$(+)\frac{h_1}{y_1}$	$(+)\frac{h_2}{y_2}$	$(-)\frac{h_3}{y_3}$	$(-)\frac{h_4}{x_4}$
versed sine of α.....	$\text{vers}\ \alpha = 1 - \cos\alpha$				
coversed sine of α...	$\text{covers}\ \alpha = 1 - \sin\alpha$				

Note the signs of the ratios in the different quadrants. These depend on the signs of x and y in the various quadrants.

The above table and figure must be memorized as a basis of future work.

The haversine is defined as follows:

$$\text{hav}\, x = \tfrac{1}{2}\, \text{vers}\, x = \tfrac{1}{2}\,(1 - \cos x).$$

By use of tables of natural haversines and their logarithms the solution of many of the problems in nautical astronomy is greatly simplified.

An angle is said to be in or to lie in that quadrant in which its terminal line lies. Thus α_1 is in the first quadrant, and α_3 is in

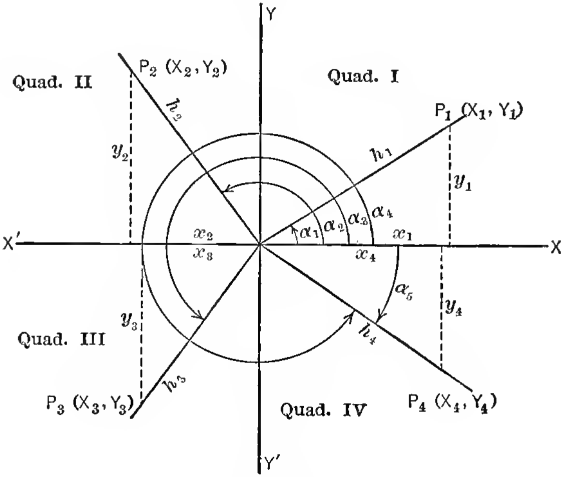


FIG. 22.

the third quadrant. Functions of negative angles are defined exactly as for positive angles. Thus $\sin \alpha_5 = y_4/h_4$, etc.

1. In what quadrant is the angle 140°? 210? 85°? 240?

2. What is the sign of each function of each angle in Ex. (1) above?

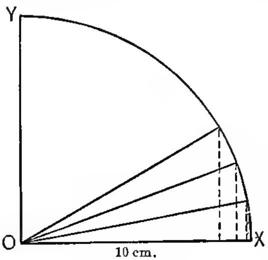


FIG. 23.

3. Draw an arc of 90° with a radius 10 cm. Divide the arc into 10° arcs. Measure the coördinates of the points of division. Divide each ordinate by the radius, 10 cm., and tabulate the quotients. Divide each abscissa by the radius and tabulate. Divide each ordinate by the corresponding abscissa. Arrange all these ratios in a table with the corresponding angles and compare them with the values given in the table of sines, cosines and tangents for the same angles, respectively, in the back of the book.

48. From the definitions of **47** and the Pythagorean theorem the following **fundamental formulas** which hold in all the quadrants are derived:

(1) $\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = \frac{x^2}{h^2} + \frac{y^2}{h^2} = \frac{h^2}{h^2} = 1. \quad \therefore \quad \cos^2 \alpha + \sin^2 \alpha = 1.$*

(2) $\frac{y}{x} = \frac{1}{\frac{x}{y}}. \quad \therefore \quad \tan \alpha = \frac{1}{\cot \alpha}$ or $\tan \alpha \cot \alpha = 1.$

(3) $\frac{h}{x} = \frac{1}{\frac{x}{h}}. \quad \therefore \quad \sec \alpha = \frac{1}{\cos \alpha}$ or $\cos \alpha \sec \alpha = 1.$

(4) $\frac{h}{y} = \frac{1}{\frac{y}{h}}. \quad \therefore \quad \csc \alpha = \frac{1}{\sin \alpha}$ or $\csc \alpha \sin \alpha = 1.$

(5) $\left(\frac{h}{x}\right)^2 - 1 = \frac{h^2 - x^2}{x^2} = \frac{y^2}{x^2}. \quad \therefore \quad \sec^2 \alpha - 1 = \tan^2 \alpha.$

(6) $\left(\frac{h}{y}\right)^2 - 1 = \frac{h^2 - y^2}{y^2} = \frac{x^2}{y^2}. \quad \therefore \quad \csc^2 \alpha - 1 = \cot^2 \alpha.$

(7) $\frac{\frac{y}{h}}{\frac{x}{h}} = \frac{y}{x}. \quad \therefore \quad \frac{\sin \alpha}{\cos \alpha} = \tan \alpha.$

The formulas in the right-hand column must be memorized.

1. Given $\sin A = 1/2$, find $\cos A$ and $\tan A$. Construct a right triangle whose hypotenuse is 2 and whose vertical leg is 1, as in Fig. 24. Calculate $x = \sqrt{2^2 - 1^2} = \sqrt{3} = 1.732$.

$$\cos A = \frac{1.732}{2} = 0.866, \text{ by the definition, } \mathbf{47}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{0.5}{0.866} = 0.577.$$

2. Given $\sec A = 3$, find $\cos A$ and $\sin A$. Construct a

* The symbol $\sin^2 \alpha$ is used for $(\sin \alpha)^2$, being more convenient. Similar notations are used for the other ratios.

right triangle whose hypothenuse is 3 and horizontal leg is 1 as in Fig. 25. Calculate $y = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2} = 2.828$.

$$\cos A = \frac{x}{h} = \frac{1}{3} = 0.333. \qquad \sin A = \frac{y}{h} = \frac{2.828}{3} = 0.9427.$$

3. Given $\sin A = \frac{2}{3}$, calculate $\cos A$, $\tan A$, and $\sec A$.

4. Given $\tan A = \frac{4}{5}$, calculate $\sin A$ and $\cos A$.

5. If one leg of a right triangle is 24 and the hypotenuse 30, find all the functions of the angle adjacent the given leg.

6. One leg of a right triangle is half the hypotenuse. Find all the functions of the angle opposite the leg.

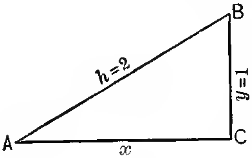


Fig. 24.

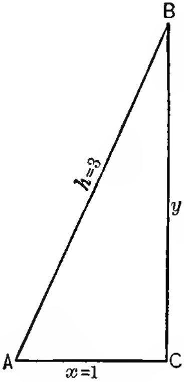


Fig. 25.

7. If $\sec\theta = 1.5$, find $\tan\theta$ and $\sin\theta$.

8. If $\tan\theta = 2.5$, find $\sin\theta$ and $\cos\theta$.

9. The hypotenuse of a right triangle is 12, the base is 8, find the sine and tangent of the angle opposite the base.

10. If the tangent of A is 1, find the secant of A and the sine of A.

In order that the student shall become sufficiently familiar with the seven fundamental formulas and their use the exercises below should be solved. The given equations are identities. Either or both sides are to be modified by various substitutions from the fundamental formulas so that both sides shall appear to be identically equal.

1. $\text{Cos}\, A \tan A = \sin A$.

Write this in the form, using Eq. 7,

$$\cos A \cdot \frac{\sin A}{\cos A} = \sin A.$$

Reducing the left side this becomes

$$\sin A = \sin A$$

which is known to be identically true for all values of A.

2. Sec A sin A = tan A. (Use equations 3, 7.)
3. Cos A csc A = cot A.
4. $(\sin A - \cos A)^2 = 1 - 2 \sin A \cos A$. (Expand and use Eq. 1.)
5. $\sin A \cot A = \cos A$.
6. $\csc A \tan A = \sec A$.
7. $\cos A/(\sin A \cot^2 A) = \tan A$.
8. $(\sin^3 A - \cos^3 A) = (\sin A - \cos A)(1 + \sin A \cos A)$.
9. $(a \cos A + b \sin A)^2 + (a \sin A - b \cos A)^2 = a^2 + b^2$.
10. $\sec^2 A + \csc^2 A = \tan^2 A + \cot^2 A + 2$.
11. $\cos A/(1 - \tan A) + \sin A/(1 - \cot A) = \sin A + \cos A$.
12. $\cot^2 A - \cos^2 A = \cos^2 A \cot^2 A$.
13. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.
14. $\sec A + \tan A = \cos A/(1 - \sin A)$.
15. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$.
16. $\cot^4 A + \cot^2 A = \csc^4 A - \csc^2 A$.
17. $(\sin A + \csc A)^2 + (\cos A + \sec A)^2 = \tan^2 A + \cot^2 A + 7$.
18. $\sin A(\tan A - 1) - \cos A(\cot A - 1) = \sec A - \csc A$.
19. $\tan^2 A - \cot^2 A = \sec^2 A \csc^2 A(\sin^2 A - \cos^2 A)$.
20. $2 \operatorname{vers} A + \cos^2 A = 1 + \operatorname{vers}^2 A$.
21. $\cos^2 A(1 + \tan^2 A) = 1$.
22. $(\sec^2 A - 1)(\csc^2 A - 1) = 1$.
23. $\tan A + \cot A = \sec A \csc A$.
24. $\sin A/\csc A + \cos A/\sec A = 1$.
25. $\sec^2 A - \sec^2 A \sin^2 A = 1$.
26. $(\tan A - 1)/(\tan A + 1) = (1 - \cot A)/(1 + \cot A)$.
27. $(\tan A + \cot A)^2 = \sec^2 A + \csc^2 A$.
28. $(\sec A - \csc A)/(\sec A + \csc A) = (\tan A - 1)/(\tan A + 1)$.
29. $(\csc A - \cot A)^2 = (1 - \cos A)/(1 + \cos A)$.
30. Show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, if $x = a \cos A$, $y = b \sin A$.

49. Functions of angles at the **quadrant limits:**

1. Sin 0° and cos 0°.

In Fig. 26 as OP moves toward OX, the angle, $\theta \to 0°$. At the same time the ordinate, $y \to 0$ and $x \to h$. We are, therefore, justified in concluding

$$\lim_{\theta \to 0°} \sin\theta = \lim_{y \to 0} \frac{y}{h} = \frac{0}{h} = 0.$$

$$\therefore \quad \mathbf{\sin 0° = 0.}$$

By similar reasoning,

$$\lim_{\theta \to 0°} \cos\theta = \lim_{x \to h} \frac{x}{h} = \frac{h}{h} = 1.$$

$$\therefore \quad \mathbf{\cos 0° = 1.}$$

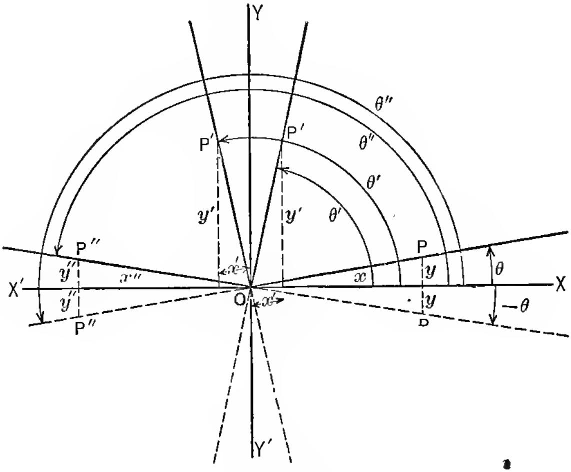


FIG. 26.
In Fig. 26 $OP = h$, $OP' = h'$, etc.

2. Sin 90° and cos 90°: By reasoning analogous to the above

$$\lim_{\theta \to 90°} \sin\theta = \lim_{y' \to h'} \frac{y'}{h'} = \frac{h'}{h'} = 1.$$

$$\therefore \quad \mathbf{\sin 90° = 1.}$$

Again,

$$\lim_{\theta \to 90^\circ} \cos\theta = \lim_{x' \to 0} \frac{x'}{h'} = \frac{0}{h'} = 0.$$

$$\therefore \quad \mathbf{\cos 90^\circ = 0.}$$

3. Sin 180° and cos 180°:

$$\lim_{\theta \to 180^\circ} \sin\theta = \lim_{y'' \to 0} \frac{y''}{h''} = \frac{0}{h''} = 0.$$

$$\therefore \quad \mathbf{\sin 180^\circ = 0.}$$

$$\lim_{\theta \to 180^\circ} \cos\theta = \lim_{x'' \to -h''} \frac{x''}{h''} = \frac{-h''}{h''} = -1.$$

$$\therefore \quad \mathbf{\cos 180^\circ = -1.}$$

Note. — It is evident that each of the limits above is the same whether the respective variables increase or decrease toward their limits.

1. Derive by a method similar to the above **sin 270° = −1, cos 270° = 0; sin 360° = 0, cos 360° = 1.**

2. From the values of the sines and cosines, by use of the fundamental formulas, obtain the values of the tangent, cotangent, secant and cosecant of 0°, 90°, 180°, 270°, 360°.

50. As was implied in **46, negative angles** and their functions must be considered. It is easily seen that for every positive angle there exists a negative angle of equal magnitude. The coördinates of P' (Fig. 27) determine the functions of $-\theta$ in the same way that the coördinates of P determine the functions of θ.

Now
$$\left|\frac{y}{h}\right| = \left|\frac{-y}{h}\right|.$$

Hence $|\sin\theta| = |\sin(-\theta)|$ and

$$\mathbf{\sin(-\theta) = -\sin\theta.} \tag{8}$$

Since x and h are the same for $-\theta$ as for θ, it follows that

$$\mathbf{\cos(-\theta) = \cos\theta.} \tag{9}$$

1. By use of the formulas of **47** derive $\tan(-\theta)$, $\cot(-\theta)$, $\sec(-\theta)$ and $\csc(-\theta)$ in terms of the same named functions of θ.

2. How can sin $(-35°)$, cos $(-20°)$ be determined from a table of sines and cosines where only positive angles are considered?

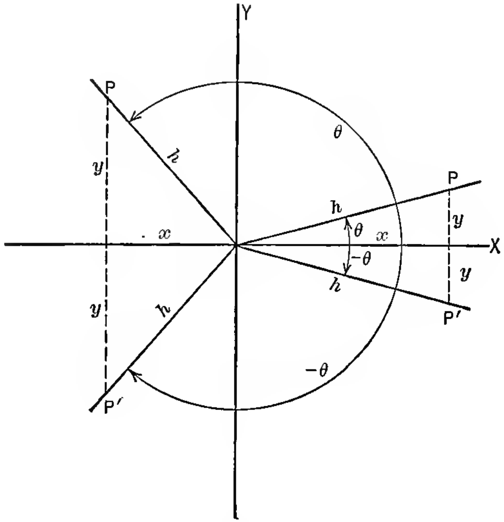


FIG. 27.

51. If for any position of OP, Fig. 28, where P is the point P (x, y) a line OQ be drawn, where Q is the point Q (x_1, y_1), where $x = y_1$, $y = x_1$, then the angle XOQ will be the complement of the angle XOP. From the definitions of **47** and the above construction it follows that $\sin XOP = \cos XOQ$ or

$$\sin \alpha = \cos (90° - \alpha), \tag{10}$$

and

$$\tan \alpha = \cot (90° - \alpha), \tag{11}$$

and

$$\sec \alpha = \csc (90° - \alpha). \tag{12}$$

These equations and the method of demonstration apply to all quadrants. These formulas will be named the **complement relations.**

1. From $\sin 30° = 0.5$, find $\cos 60°$; From $\sec 45° = 1.414$, find $\csc 45°$.

2. From $\sin 60° = 0.866$, find $\cos 30°$; From $\tan 45° = 1$, find $\cot 45°$.

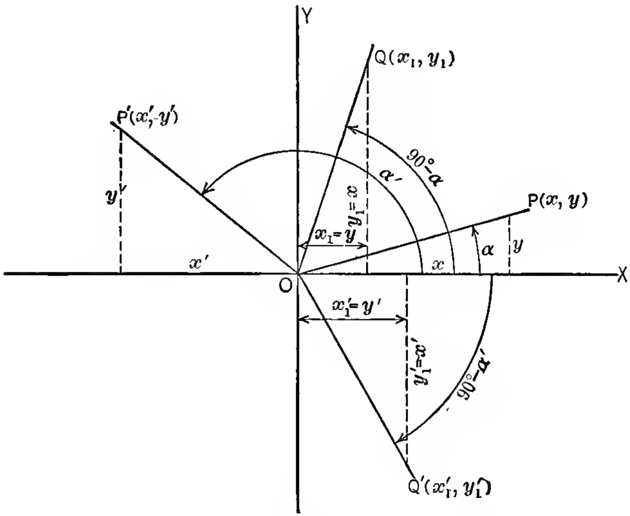


FIG. 28.

3. From $\cos 120° = -0.5$, find $\sin(-30°)$.

4. From $\tan 60° = \sqrt{3}$, find $\cot 30°$.

5. The legs of a right triangle are $x = 5$, $y = 7$. Calculate all the functions of the angle opposite the longest leg.

6. By use of formulas of **51** obtain all functions of the angle opposite the shortest leg from the values calculated in Ex. 5.

52. To reduce the **functions** of **any angle** to the same named functions of an angle less than 90°.

1. Consider $90° < \alpha < 180°$. Construct the triangles POM and $P'OM'$, Fig. 29, so that

$h = h'$, $\alpha' = A$, $-x = x'$, then $y = y'$, $A = (180° - \alpha) = \alpha'$.

Then $$\sin \alpha = y'/h' = y/h = \sin \alpha'.$$

$$(13) \qquad \therefore \quad \sin \alpha = \sin (180° - \alpha).$$

Also, since $-x = x'$,

(14) $$\cos \alpha = \frac{x'}{h'} = -\frac{x}{h} = -\cos (180° - \alpha).$$

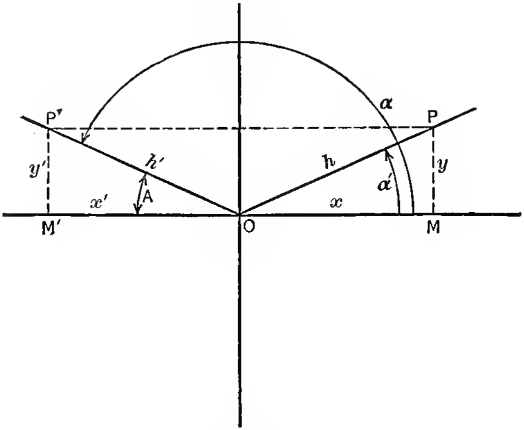


FIG. 29.

2. Consider $180° < \alpha < 270°$. Construct the triangles POM and $P'OM'$ (Fig. 30) so that $A = \alpha'$, $y = -y'$, $x = -x'$, then $h = h'$,

and $\alpha' = \alpha - 180°$. Evidently,

$$\sin \alpha = y/h = -y'/h' = -\sin \alpha'.$$

(15) $$\therefore \quad \sin \alpha = -\sin (\alpha - 180°).$$

Also, $$\cos \alpha = x/h = -x'/h' = -\cos \alpha'.$$

(16) $$\therefore \quad \cos \alpha = -\cos (\alpha - 180°).$$

3. Consider $270° < \alpha < 360°$. Construct the triangles POM and $P'OM$ (Fig. 31) so that

$$A = \alpha', \quad y = -y', \quad x = x', \quad \text{then } h = h',$$

and $\alpha' = 360 - \alpha = A$. Evidently,

$$\sin \alpha = y/h = -y'/h' = -\sin \alpha'.$$

(17) $$\therefore \quad \sin \alpha = -\sin (360° - \alpha),$$

and $$\cos \alpha = x/h = x/h' = \cos \alpha'.$$

(18) $$\therefore \quad \cos \alpha = \cos (360° - \alpha).$$

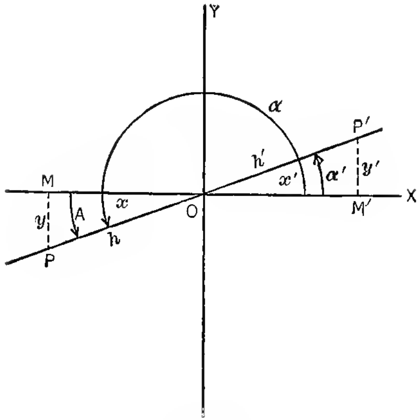


FIG. 30.

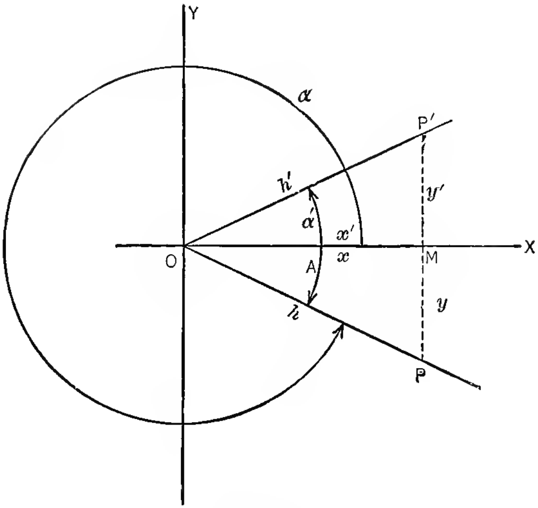


FIG. 31.

4. Consider $90° < \alpha < 180°$ and $\alpha - 90°$. Construct the triangles POM and $P'OM'$ so that

$$y = x', \quad x = -y', \quad \text{then } \alpha' = \alpha - 90°.$$

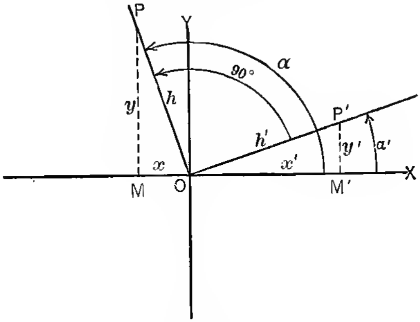


FIG. 32.

Then $\sin \alpha = y/h = x'/h' = \cos \alpha'$.

(20) $\therefore$ $\sin \alpha = \cos (\alpha - 90°)$.

Again $\cos \alpha = x/h = -y'/h' = -\sin \alpha'$.

(21) $\therefore$ $\cos \alpha = -\sin (\alpha - 90°)$.

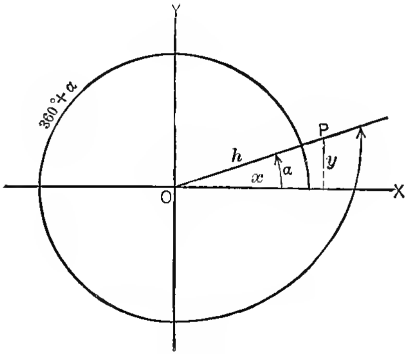


FIG. 33.

As an exercise let the student derive from these results the tangent of α, under all the above cases.

5. Consider $360 < \alpha$. It is evident that if 360° be added to any angle α, the terminal line will coincide with that of α. It follows that the values of the defining ratios of the functions of $\alpha + 360°$ will be identical with those of α. It is evident then that

$$\sin \alpha = \sin (360 + \alpha), \tag{22}$$

$$\cos \alpha = \cos (360 + \alpha), \tag{23}$$

and similarly for all the functions. It is equally evident that 360° may be subtracted from α without affecting the values of the functions of α.

53. Addition theorems. — The formulas

$$\sin (A + B) = \sin A \cos B + \cos A \sin B, \tag{1}$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \tag{2}$$

are known as the addition theorems for the sine and cosine respectively.

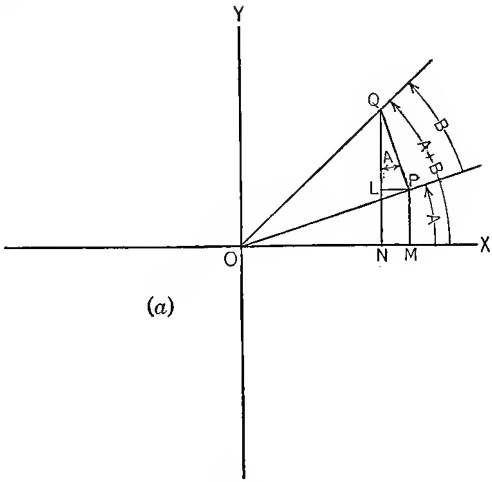


FIG. 34.

To prove (1), consider either position of Q (Fig. 34a and 34b), where $0° < A < 90°$ and $0° < B < 90°$. Then $A + B < 180°$.

$$\sin (A + B) = NQ/OQ = MP/OQ + LQ/OQ.$$

But $MP = OP \sin A$ and $LQ = PQ \cos A$, since angle $LQP = A$. Substituting these values in the above equation gives

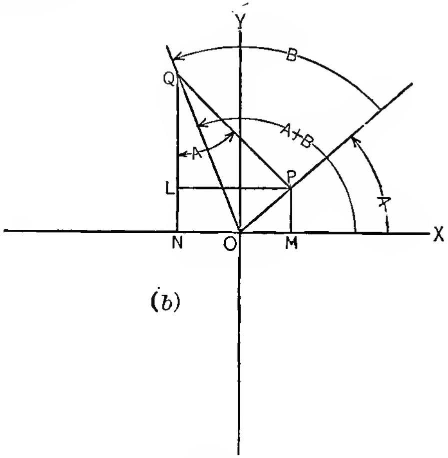
$$\sin (A + B) = \frac{OP}{OQ} \sin A + \frac{PQ}{OQ} \cos A.$$

But $$\frac{OP}{OQ} = \cos B \quad \text{and} \quad \frac{PQ}{OQ} = \sin B.$$

Substituting these values in the last equation gives

(24) $$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

This is equation (1).



(*b*)

FIG. 34.

To prove (2) consider the same figure:

$$\cos (A + B) = \frac{ON}{OQ} = \frac{OM}{OQ} - \frac{NM}{OQ} = \frac{OM}{OQ} - \frac{LP}{OQ}.$$

But $$OM = OP \cos A \quad \text{and} \quad LP = QP \sin A.$$

Substituting these values in the above equation gives:

$$\cos (A + B) = \frac{OP}{OQ} \cos A - \frac{QP}{OQ} \sin A.$$

But $$\frac{OP}{OQ} = \cos B \quad \text{and} \quad \frac{PQ}{OQ} = \sin B.$$

Substituting these values in the last equation gives

(25) $$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

This is equation (2).

These proofs may be extended to **angles of any magnitude** by the use of **52.** For $\alpha > 90°$, $\sin \alpha$ can be expressed in terms of $\sin \alpha'$ where $\alpha' < 90°$. Therefore, when $A > 90°$ and $B > 90°$ and $A + B > 180°$ the sines and cosines of A and B can be expressed in terms of like-named functions of angles less than

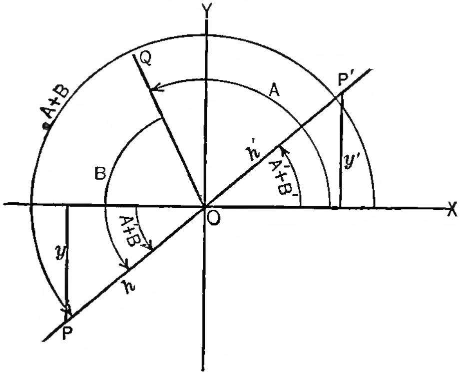


FIG. 35.

90°, say A' and B', respectively. Corresponding to $A + B >$ 180° there will be $A' + B' < 180°$. Then $\sin (A + B)$ can be expressed in terms of $\sin (A' + B')$. The generality of the equations (1), (2) for all angles is inferred. The proof may also be generalized. If

$$90° < A < 180°, \quad 90° < B < 180°, \quad 180° < A + B < 360°.$$

Then in the figure

$$\sin (A + B) = -\sin (A' + B') = y'/h' = -y/h,$$

$$A + B - 180° = A' + B' = (A - 90°) + (B - 90°),$$

where $0° < A' < 90°, \quad 0° < B' < 90°, \quad 0° < A' + B' < 180°.$

$$\therefore \quad \sin (A + B) = -\sin (A' + B'),$$

This case has been proved, since $A' + B' < 180$.

It is known that:

$$\sin(A - 90°) = -\cos A = \sin A',$$
$$\cos(A - 90°) = \sin A = \cos A',$$
$$\sin(B - 90°) = -\cos B = \sin B',$$
$$\cos(B - 90°) = \sin B = \cos B'.$$

From these

$$\sin(A+B) = -\sin(A'+B') = -((-\cos A)\sin B + (-\cos B)\sin A).$$
$$\therefore \quad \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

This establishes the theorem for all values of A and B so long as $A + B < 360°$. As an exercise let the student establish the equation (2) for the same values of A, B.

In a similar manner the proof may be extended to the case of $A + B > 360°$. The addition theorems are general.

The addition theorems for the other functions can be easily deduced from those of the sine and cosine by algebraic methods. Thus for $\tan(A + B)$, write

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing numerator and denominator of the last fraction by $\cos A \cos B$ gives

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}.$$

(26) $$\therefore \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

for all values of A and B.

1. By use of the definitions of functions of negative angles derive from (24), (25), (26),

(27) $$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

(28) $$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

(29) $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

2. If $\sin A = 0.5$ and $\sin B = 0.25$, find $\sin(A+B)$, $\cos(A+B)$, $\tan(A+B)$, $\sin(A-B)$, $\cos(A-B)$, $\tan(A-B)$.

Note. — Cos A and $\cos B$ must first be found. Then substitute in the addition theorems.

3. By use of the addition theorems derive:

(30) $$\sin(90^\circ + A) = \cos A.$$
(31) $$\cos(90^\circ + A) = -\sin A.$$
(32) $$\sin(90^\circ - A) = \cos A.$$
(33) $$\cos(90^\circ - A) = \sin A.$$
(34) $$\sin(180^\circ - A) = \sin A.$$
(35) $$\cos(180^\circ - A) = -\cos A.$$
(36) $$\sin(A - 90^\circ) = -\cos A.$$

4. By use of the addition theorem for $A = B$ or $A + A = 2A$, derive

(37) $$\sin 2A = 2\sin A\cos A,$$
(38) $$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A.\end{aligned}$$

5. By use of the addition theorems with $A + 2A = 3A$, derive expressions for $\sin 3A$ and $\cos 3A$ in terms of $\sin A$ and $\cos A$.

6. Given $\sin A = 0.6$, find $\sin 2A$, $\cos 2A$, $\sin 3A$, $\cos 3A$.

7. Given $\sin A = \frac{1}{3}$, $\sin B = \frac{2}{3}$, find $\sin(A+B)$ $\cos(A+B)$ $\tan(A+B)$.

8. Given $\cos 2A = 0.866$, find $\cos A$ and $\sin A$.

Note. — Write $\cos 2A = 2\cos^2 A - 1 = 0.866$, and solve the equation for $\cos A$.

9. Given $\tan 2A = 1.5$, find $\tan A$ and $\cos A$.

10. Given $\tan A = 0.8$, find $\tan(180 + A)$, $\tan(180 - A)$.

11. From $\sin 45^\circ = 0.7071$, find $\sin 22^\circ 30'$.

54. In the **solution** of **problems** an **unknown** angle often occurs through one of its functions, say a sine or cosine. It is then necessary to solve the equation to obtain the angle. To do this the function of the unknown angle is regarded as the unknown quantity in the equation instead of the angle itself.

When the function has been found the corresponding angle can be found from the table. To illustrate, suppose there is given $1 + \sin A = \frac{3}{2}$, from which to find A. Transposing and simplifying give

$$\sin A = \tfrac{1}{2}.$$

From the table, A is found to be 30°. By use of formulas of **52** $\sin 30° = \sin (180° - 30°) = \sin 150°$. Hence 150° is another value of A which satisfies the equation.

Instead of finding A in degrees it may be desirable to use A merely as an angle belonging to its sine, $\frac{1}{2}$. For this purpose several notations are used. Thus

$$A = \text{angle whose sine is } \tfrac{1}{2}, \quad A = \text{arc} \sin \tfrac{1}{2}, \quad A = \sin^{-1} \tfrac{1}{2},$$

(read inverse sine $\frac{1}{2}$) all mean the same thing. Either of the equations

$$A = \text{arc} \sin x$$

and

$$x = \sin A$$

implies the existence of the other. Similar notations apply to all the functions of the angle.

1. What is the value of arc $\sin \frac{1}{3}$? Of arc sin 1? Of arc $\sin (-\frac{1}{2})$?

2. What is the value of arc tan 1? Of arc cos 0? Of arc sec 3?

3. What is the value of arc $\sin \frac{1}{2}$ + arc $\cos \frac{1}{2}$? Of arc tan 1 + arc cot 1?

4. What is the value of $\sin (\text{arc} \sin \frac{1}{2})$? Of $\cos (\text{arc} \sin \frac{1}{2})$?

5. What is the value of $\sin \left(\text{arc} \sin \frac{1}{2} + \text{arc} \sin \frac{1}{2}\sqrt{3}\right)$? By the addition theorem this may be expressed as $\sin (A + B) = \sin A \cos B + \cos A \sin B$, where $A = \text{arc} \sin \frac{1}{2}$ and $B = \text{arc} \sin \frac{1}{2}\sqrt{3}$. Then

$$\sin \left(\text{arc} \sin \left(\tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} \cdot \tfrac{1}{2}\sqrt{3}\right)\right) = \sin (\text{arc} \sin 1) = 1.$$

6. If $x = \sin A$ and $y = \sin B$, show that $(A + B) = \text{arc} \sin \left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right)$.

7. Find x from $\text{arc} \sin 0.5 = x$.

8. Find x from arc tan $15 = x$.

9. Solve for x in the equation arc sin $x =$ arc cos $(x - \frac{1}{2})$.

Note. — Take the sine of both sides first.

10. Solve for x in $\sin(x - 25°) = 0.6$.

Note. — Expand the left side by the addition theorem, and use **48**.

11. Solve for x in arc tan $x =$ arc sec $x - 45°$.

12. Solve for x in arc tan $x +$ arc cot $x = 90°$.

13. Solve for x in tan (arc tan x) $= 1$.

55. One of the **chief uses** of the trigonometric functions is found in the solution of problems relating to triangles. All questions relating to right triangles can be solved by direct use of the definitions of **47**, for the first quadrant. For this purpose the definitions for acute angles can be modified as follows: Consider the $\triangle ABC$ in the figure, C being the right angle. Then

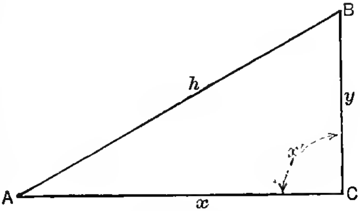


FIG. 36.

$$\sin A = \frac{y}{h} = \frac{\text{side opposite } A}{\text{hypotenuse}},$$

$$\cos A = \frac{x}{h} = \frac{\text{side adjacent } A}{\text{hypotenuse}},$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite } A}{\text{side adjacent } A}.$$

Similar relations hold for the angle B.

56. General directions for solving problems relating to triangles:

1. Make a fairly accurate freehand diagram from the given conditions.

2. Mark all known measurements on the diagram. Note the position and relations of the unknowns.

3. Select a formula which will contain one of the unknowns and the knowns.

4. Substitute the values of the knowns in the formula and solve the resulting equation for the unknown.

5. When no formula fits the case directly, designate auxiliary unknowns by symbols. Formulate as many equations as unknowns and solve the system of equations simultaneously as in algebra.

1. Find the height of a tree, having given the angle of elevation, $A = 35°$, and the distance $AB = 125'$.

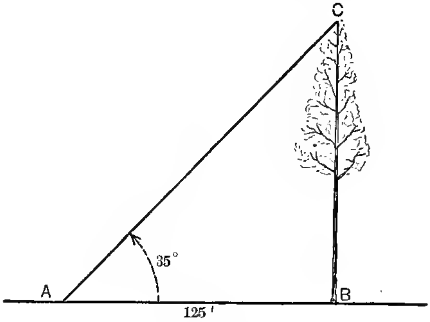


FIG. 37.

Note. — The given measurements are the angle A and the side adjacent. The desired measurement is the side opposite A. We select, therefore, the tangent ratio. Write

$$\tan A = \frac{CB}{AB}.$$

Substituting values

$$\tan 35° = \frac{CB}{125}.$$

From the table

$$0.7002 = \frac{CB}{125}.$$

Solving for CB,

$$CB = 87.53.$$

2. From the ends of a line AB perpendiculars are dropped on MN meeting MN in C and D respectively. The angle

between MN and AB is 47° 30′. The line AB is 565 units long. Find CD.

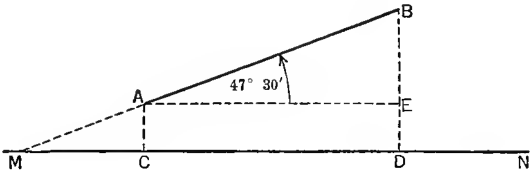


FIG. 38.

Note. — CD is called the projection of AB on MN. The angle AEB is called the angle of projection. See Geometry.

3. Find the projection of a line 50′ long on a line at an angle of 60° with it.

4. What are the projections on the axes of a line 120′ long which is inclined 36° with the x-axis? The lengths of AB and CD in the diagram are wanted.

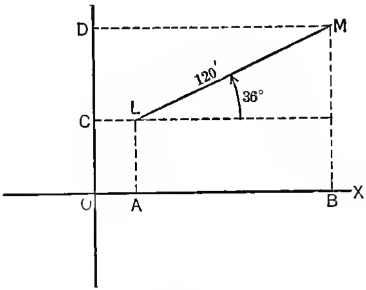


FIG. 39.

5. A plane surface may be projected on a plane inclined to it in a manner exactly similar to the projection of a line on another line.

Find the area of the projection of a rectangle 30′ x 60′ on a plane inclined 75° to it.

6. A roof is 30′ x 20′ and is inclined 37° to the horizontal. How many sq. ft. of floor does it cover?

7. What is the area of the projection of a circle of 10′ radius on a plane inclined to it at an angle of 27°?

8. A circle has a radius of 10′. A chord of the circle is 15′ long. How far is the chord from the center?

9. Find the perimeter of a regular pentagon inscribed in a circle of radius 5″.

10. Find BD in the diagram if $AC = 100'$, angle $A = 25°$ and angle $C = 30°$.

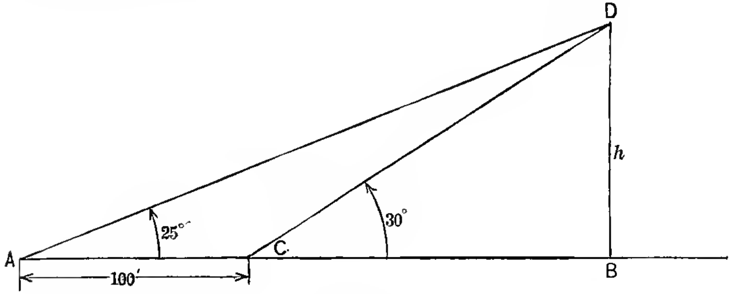


FIG. 40.

Note. — Introduce the auxiliary unknown x for CB and formulate two equations. Then eliminate x.

11. What is the eastward component of a force of 105 lbs. acting in a direction 30° east of due north? The length OM represents the eastward component. Find the northward component.

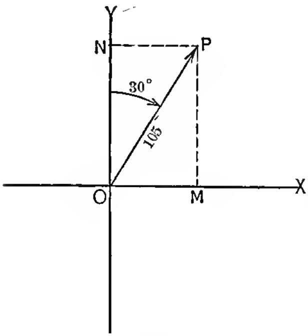


FIG. 41.

12. In the triangle ABC, $A = 42°$, $AB = 125$, $AC = 150$. Find the altitude on the side AC and the area of the triangle. From this problem determine a formula for the area of any triangle when two sides and their included angle are given.

13. By drawing an equilateral triangle and its altitude derive values of sin 30°, cos 30°, sin 60°, cos 60°.

14. By considering a square and its diagonal derive values of sin 45°, cos 45°.

15. A travels north 50 mi., then 37° east of north a distance 75 mi., then 10° west of south 125 mi. Find the length and direction of the line from the starting point to his final position.

Find the distance east or west he traveled and the distance north or south, from the starting point.

57. The sine law of triangles. — From the figure it is seen that

$$h = c \sin A \text{ and } h = a \sin C.$$

$$\therefore \quad c \sin A = a \sin C$$

and
$$\frac{c}{\sin C} = \frac{a}{\sin A}.$$

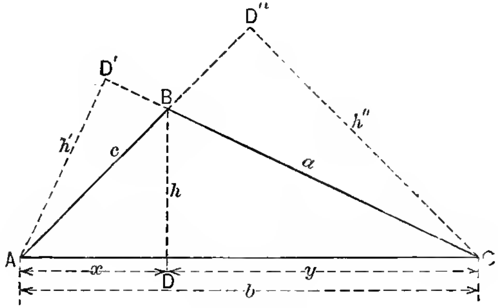


FIG. 42.

By drawing altitudes from the other vertices the equations

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

and
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

can be derived in exactly the same way as above.

$$(39) \qquad \therefore \quad \frac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin \boldsymbol{C}}.$$

These equations are known as the sine law. Its use is indicated when a side and the angle opposite are among the parts to be considered.

58. The cosine law of triangles. — From the figure of the last section,

$$c^2 = h^2 + x^2 = h^2 + (b - y)^2 = h^2 + b^2 - 2\,by + y^2.$$

But $h = a \sin C$, $y = a \cos C$. Substituting in the above equation

$$c^2 = b^2 - 2\,ab \cos C + a^2 \sin^2 C + a^2 \cos^2 C$$

(40) or $\boldsymbol{c^2 = a^2 + b^2 - 2\,ab \cos C.}$

By using the other sides of the triangle as bases in turn, the following are derived in the same way:

(40a) $\boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos A.}$

(40b) $\boldsymbol{b^2 = a^2 + c^2 - 2\,ac \cos B.}$

59. Example of the use of the sine law. — Given $A = 15° \, 19'$, $C = 72° \, 44'$, $c = 250.4$, of the triangle ABC, to find the remaining parts. Immediately

$$B = 180 - (A + C) = 91° \, 57'.$$

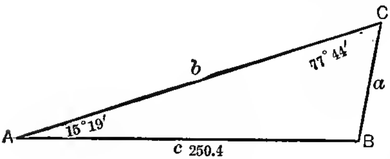


FIG. 43.

By the sine law,

$$\frac{a}{\sin 15° \, 19'} = \frac{250.4}{\sin 72° \, 44'},$$

or $$a = \frac{250.4 \sin 15° \, 19'}{\sin 72° \, 44'}.$$

In logarithms this is

$$\begin{aligned} \log a &= \log 250.4 + \log \sin 15° \, 19' - \log \sin 72° \, 44' \\ &= \quad 2.3986 \\ &\quad + \underline{9.4218 - 10} \\ &\quad\ 11.8204 - 10 \\ &\quad - \underline{9.9800 - 10} \\ &= \quad 1.8404 \end{aligned}$$

$\therefore \quad a = 69.25.$

Again, to find b, using the sine law,

$$\frac{\sin 72° \, 44'}{250.4} = \frac{\sin 91° \, 57'}{b}.$$

Solving as above, $b = 262.0.$

1. Student check the result with natural functions, using the slide rule.

2. Construct the triangle to scale 50 to 1″ from the given data and then measure the unknown parts. Check with the calculated values.

60. Example of the use of the cosine law. — Given $a = 1686$, $b = 960$, $C = 128° 04'$, to find the other parts.

By the cosine law

$$c^2 = 1686^2 + 960^2 - 2 \cdot 1686 \cdot 960 \cdot \cos 128° 4'.^*$$

Whence $$c = 2400.$$

The values of A and B may now be found by the sine law.

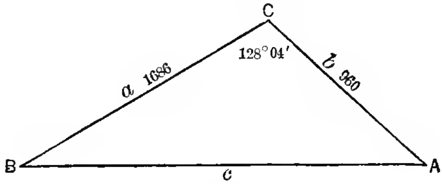


FIG. 44.

1. Find AB, the distance across a river, from the data given in the diagram. C is a point on top of a hill, AB and CD are horizontal lines. $BC = 500'$, angle $DCA = 15°$, angle $CBE = 20°$.

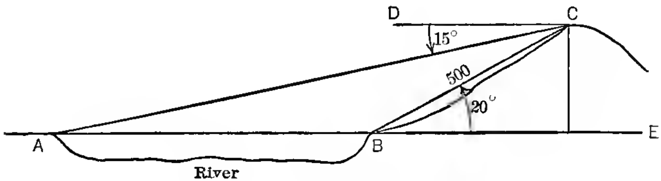


FIG. 45.

2. Two trains leave a station at the same time on straight tracks inclined to each other at an angle of 35°. Train A travels 25 mi. per hr., train B travels 35 mi. per hr. How far apart are the trains at the end of 3 hrs.?

* Note that cos 128° 4′ is negative.

3. Find AB from the measurements given below. $CD = 1000'$, $ACD = 120°$, $DCB = 30°$, $ADC = 33°$, $CDB = 105°$.

4. Use the cosine law to find the angles of the triangle ABC, if $a = 75.8$, $b = 64.2$, $c = 81.9$.

5. A tower stands at the top of a slope inclined 20° with the horizontal. From a point 800′ down the slope from the tower the angle subtended by the tower is 5° 25′. Find the height of the tower.

6. The tripod of a camera stands on a hillside. One leg is 3′ long, the other two legs are each 5′ and set on the ground at the same level. The three legs make equal angles with each other successively around. These angles are each 38°. Find the distance between the feet of the legs.

61. Conversion formulas. — The cosine law does not admit of use with logarithms. For this reason another law, derived from the sine law, is used when it is desired to employ logarithmic calculations. This law is known as the tangent law. Before the tangent law can be given some formulas must be developed.

From the addition theorems we have:

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B.$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B.$
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B.$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B.$

Add (2) to (1); subtract (2) from (1); add (4) to (3); subtract (4) from (3), and obtain the following equations in order.

5. $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$
6. $\sin(A + B) - \sin(A - B) = 2 \cos A \sin B.$
7. $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B.$
8. $\cos(A + B) - \cos(A - B) = -2 \sin A \sin B.$

In the last four equations substitute $A = \dfrac{P + Q}{2}$ and $B = \dfrac{P - Q}{2}$. There results:

(41) $$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

(42) $$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}.$$

(43) $$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

(44) $$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}.$$

62. The tangent law. — By the sine law

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Taking this proportion by composition and division gives

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

Applying (41), (42), to the right side gives, by use of **48**,

(45) $$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

Two similar equations using the other parts of the triangle are obtained in a similar manner. This equation is called the tangent law for triangles. This law applies directly to the case that was solved by the cosine law in **60**. That problem will now be solved by the tangent law. Write

$$\frac{1686 + 960}{1686 - 960} = \frac{\tan \frac{1}{2}(180^\circ - 128^\circ\, 04')}{\tan \frac{1}{2}(A-B)},$$

since $\frac{1}{2}(A+B) = \frac{1}{2}(180^\circ - C)$ in any triangle. Solving the above equation for $\tan \frac{1}{2}(A-B)$,

$$\tan \frac{1}{2}(A-B) = \frac{726 \cdot \tan 25^\circ\, 58'}{2646}.$$

Applying logarithms to this equation,

$$\log \tan \frac{1}{2}(A-B) = \log 726 + \log \tan 25^\circ\, 58' - \log 2646.$$

Substituting the values

$$\log \tan \frac{1}{2}(A-B) = 9.1258 - 10.$$

Whence $$\frac{1}{2}(A-B) = 7^\circ\, 37'.$$

Now $\frac{1}{2}(A+B)+\frac{1}{2}(A-B)=A=25^\circ\,58'+7^\circ\,37'=33^\circ\,35'$,

and $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B=25^\circ\,58'-7^\circ\,37'=18^\circ\,21'$.

Having the angles A and B, the remaining side can be found by the sine law.

63. Ambiguous case. — When the given parts of a triangle are two sides and an angle opposite one of these sides any one of the following results may occur:

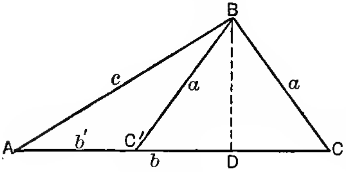


FIG. 46.

Let a, c, A be the given parts and let h be the altitude from B.

1. $a < h$, where $h = c \sin A$. Under these circumstances the triangle cannot be constructed.

2. $a = h$, where $h = c \sin A$. There is, under these circumstances, one triangle, a right triangle with the right angle at D.

3. $a > h$ and $a < c$, where $h = c \sin A$. Under these conditions there will be two triangles, ABC and ABC'.

Note. — $C' + C = 180^\circ$. Hence $\sin C' = \sin C$. In solving this case the value of C will be obtained from its sine. Since there are two angles each less than 180° having the same sine, a choice must be made between C' and C. This choice must be based on other data in the problem. When no conditions are given for making a choice both solutions must be given.

Note. — Whenever in solving a problem a result is obtained that implies that the sine or cosine of an angle is greater than 1, the problem is either impossible or an error has been made in the calculations.

(*a*) *Example.* — Given $a = 250$, $A = 42^\circ\,12'$, $c = 600$, to find the remaining parts of the triangle ABC. By the sine law,

$$\frac{\sin C}{600}=\frac{\sin 42^\circ\,12'}{250}.$$

Solving gives

$$\log \sin C = 10.2075 - 10.$$

This is equivalent to $\sin C > 1$, which is impossible. This means $250 < h$ and the triangle cannot be constructed.

(*b*) *Example.* — Given $c = 254.3$, $a = 396.8$, $A = 94° 29'$, to find the remaining parts of the triangle ABC.

By the sine law

$$\frac{396.8}{\sin 94° 29'} = \frac{254.3}{\sin C}.$$

Solving,

$$\log \sin C = 9.8054 - 10.$$
$$\therefore \quad C = 39° 43' \text{ or } 140° 17'.$$

Since an obtuse angle occurred in the given data, only the smaller value of C can be used.

(*c*) *Example.* — Given $a = 250$, $c = 300$, $A = 42° 12'$, to find the remaining parts of the triangle ABC. By the sine law,

$$\frac{\sin C}{300} = \frac{\sin 42° 12'}{250}.$$

Solving,

$$\log \sin C = 9.9064 - 10.$$
$$\therefore \quad C = 53° 43' \text{ or } 126° 17'.$$

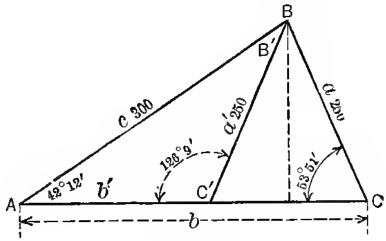


FIG. 47.

In this case either value of C will satisfy the problem and there are two possible triangles. The other parts of each triangle can be found.

64. Double and half-angle formulas. — In the addition theorems put $B = A$. Then

(46) $\quad \sin (A + A) = \sin 2A = 2 \sin A \cos A.$

47. $\cos(A+A) = \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

From **47**, $$2\sin^2 A = 1 - \cos 2A$$

or (1) $$\sin A = \sqrt{\frac{1-\cos 2A}{2}}.$$

From **47** again, $$2\cos^2 A = 1 + \cos 2A$$

or (2) $$\cos A = \sqrt{\frac{1+\cos 2A}{2}}.$$

In (1) and (2) put $A = \frac{P}{2}$ and obtain:

(48) $$\sin\frac{1}{2}P = \sqrt{\frac{1-\cos P}{2}}.$$

(49) $$\cos\frac{1}{2}P = \sqrt{\frac{1+\cos P}{2}}.$$

As an exercise derive $\tan\frac{1}{2}P$, $\cot\frac{1}{2}P$, $\sec\frac{1}{2}P$, $\csc\frac{1}{2}P$.

65. Angles of a triangle in terms of its sides. — By the cosine law:

(1) $$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

Add 1 to both sides,

(2) $$1 + \cos A = \frac{b^2+2bc+c^2-a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Subtract both sides of (1) from 1,

(3) $$1 - \cos A = \frac{a^2-(b^2-2bc+c^2)}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}.$$

Substituting these values in (48), (49), **64**, gives:

(50) $$\sin\frac{1}{2}A = \sqrt{\frac{(a-b+c)(a+b-c)}{4bc}} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

(51) $$\cos\frac{1}{2}A = \sqrt{\frac{(a+b+c)(b+c-a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}},$$

where $a + b + c = 2s$. Note the arrangement of letters in the last two equations.

A is any angle of the triangle:

1. Derive $\tan \frac{1}{2} A$ from (48), (49). $\operatorname{Tan} \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

2. Find the angles of the triangle ABC if $a = 45$, $b = 55$, $c = 66$.

66. Radius of circle inscribed in a triangle ABC. Call $a + b + c = 2s$. By the geometry of the figure:

$$A2 + B3 + C1 = s$$

or

$$A2 = s - B3 - C1 = s - B3 - C3$$
$$= s - a.$$

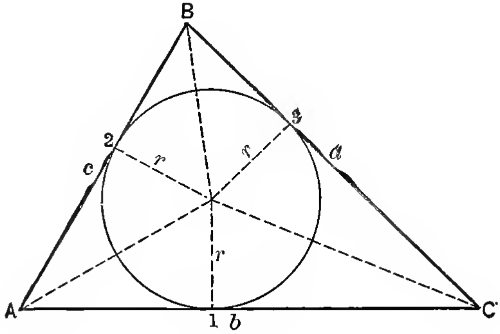


FIG. 48.

Now

$$\tan \frac{1}{2} A = \frac{r}{A2} = \frac{r}{s-a}$$

or

(52) $$\boldsymbol{r = (s - a) \tan \tfrac{1}{2} A,}$$

where a is any side and A the opposite angle of the triangle.

67. Radius of circle circumscribed about a triangle ABC. In Fig. 49,

$$\text{angle } A' = \text{angle } A,$$
$$\text{angle } ABC = 90°.$$

Therefore $$\sin A = \sin A' = \frac{BC}{A'C} = \frac{BC}{2R} = \frac{a}{2R}$$

and

(53) $$R = \frac{a}{2 \sin A},$$

where A is any angle of the triangle and a the opposite side.

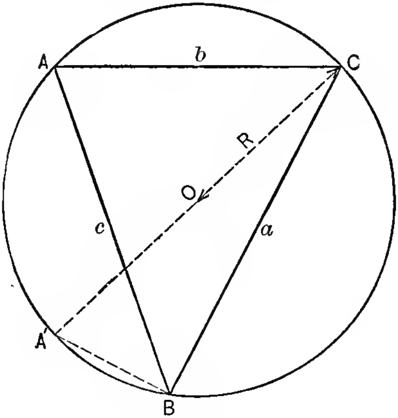


FIG. 49.

68. Circular measure of angles. — We are accustomed to measure angles in degrees. It is often convenient to measure angles with another unit. This unit is called the radian and is defined by the equation,

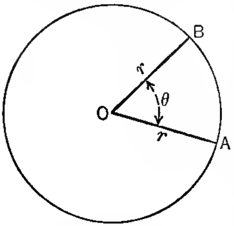


FIG. 50.

(54) $$\text{arc } \boldsymbol{AB} = \boldsymbol{r} \cdot \boldsymbol{\theta},$$

where r is the radius of the arc AB and θ is the angle in radians. Since an arc equal to the circumference has the same measure as a 360° angle at the center the relation between the degree and the radian is easily obtained.

By (54), $\theta = \dfrac{\text{arc}}{r}$ in radians, that is, the arc divided by its radius gives the measure of the subtended angle in radians.

Therefore $$\frac{\text{arc (circumference)}}{r} = \frac{2\pi r}{r} = 2\pi$$

(55) and $$\mathbf{2\,\pi \text{ radians} = 360°.}$$

$$1 \text{ radian} = \frac{360°}{2\pi} = 57°\,17'\,45''-$$

The following table of equivalents is easily derived and will be used in work to follow. It would be well to memorize this table.

Radians.	Degrees.
2π	360
π	180
$\pi/2$	90
$\pi/3$	60
$\pi/4$	45
$\pi/6$	30

1. The radius of a circle is 12. The angle at the center is $45° = \dfrac{\pi}{4}$ radians. Find the arc intercepted on the circumference.

By (54), $$\text{arc} = 12 \cdot \frac{\pi}{4} = 3\pi.$$

2. Find the arc subtended by a chord 4′ from the center of a circle of radius 10′.

The cosine of half the angle subtended by the chord is $\frac{4}{10} = 0.4$. From this the angle is found from the tables to be 23° 35′ or 0.41+ radians. Now by (52),

$$\text{arc} = 10 \times .41+ = 4.1+, \text{ ft.}$$

3. In a circle of radius 12″ a line 10″ from the center is drawn. What is the length of the arc cut off? (First find the angle subtended by the chord, then apply Eq. (52.))

4. The radius of a circle is 15, a chord is drawn cutting off a segment of altitude 3. Find the area of the segment.

5. A carriage wheel is 4′ in diameter, the carriage is traveling 10 mi. per hr. Find the number of revolutions per min. and the number of radians per sec. turned through by the wheel.

6. The arc through which a pendulum swings is 4″. The length of the pendulum is 39.4″. Find the angle of swing in radians and in degrees.

7. An arc is 10″ and the angle measured at the center is $1\frac{1}{2}$ radians. Find the radius of the circle.

8. A horizontal tank 6′ in diameter and 30′ long is filled to a depth of 2′. Find the number of gallons in the tank. See Chap. II (51).

68a. Mil, a unit of angular measure: Just as a central angle standing on an arc that is $\frac{1}{360}$ of the circumference of a circle is one degree, so also a central angle standing on an arc that is $\frac{1}{6400}$ of the circumference is **one mil.** Since by this definition there are 6400 mils at the center of a circle, it follows that the length of the arc that subtends one mil is

$$\frac{\text{circumference}}{6400} = \frac{2\,\pi r}{6400} = \frac{6.283\,r}{6400} = .0009816\,r.$$

1. In military work it is common to speak of a mil as approximately equal to an arc of one foot at a distance of 1000 feet or one yard at a distance of 1000 yards. Give reason for this statement.

2. In a table of natural sines of angles, expressed in mils, is the following:

Angle in mils	64	1600	2400	3180
sine	.0628	1	.6071	.0194

Check this table by converting mils to degrees and then use table of natural sines.

3. From problem 2 find the sine of 800 mils.

69. Explanatory definitions. — (*a*) In surveying the direction of a line is usually given by giving the angle which it makes with a north and south line. If the line runs north and east its

direction is designated by N. $\theta°$ E., where $\theta°$ is the angle between a line running due north from the starting point and the line of sight from same point. If the line runs in a northwesterly direction its direction is designated by N. $\theta°$ W. Similar notations apply to directions southeasterly and southwesterly.

(*b*) The angle of elevation of a point is the angle between a horizontal line and a line from the observer to the observed point, above the level of the observer.

(*c*) The angle of depression of a point below the observer is the angle between a horizontal line and a line from the observer to the point observed. In the figure, θ is the angle of elevation of B from A, and θ' is the angle of depression of C from A.

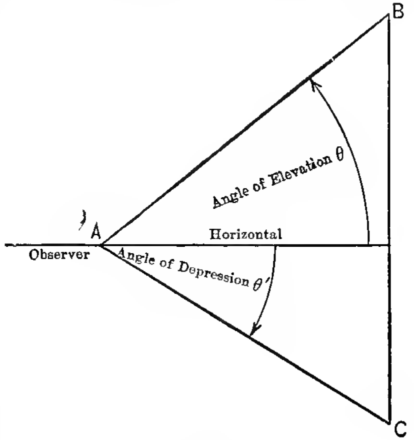


Fig. 51.

PROBLEMS.

1. The distance between two points measured on a slope of 5° 42′ with the horizontal is 210.3′. Find the horizontal distance between the points.

2. The distance between the points A and B measured on the horizontal is 388.0′. The bearing from A to B is N. 30° E. How far is B north and east of A?

3. A surveyor sets his instrument over a stake at A and reads the bearing to another stake at B, S. 22° 10′ E. The distance from A to B is 1142.1′ measured on an upward slope of 12° 21′. Find how far B is north or south of A. How far is B east or west of A? How far is B above or below A? (Save results.)

4. Compute similarly the position of C with reference to B when the bearing from B to C is S. 37° 30′ W. and the distance 843.7′ measured on a downward slope of 10° 25′. (Save results.)

5. Using the results of (3) and (4) compute the position of C with respect to A.

6. It is desired to drive a straight passage from A to B in a mine. A is known to be 3500′ north and 2200′ west of a third point C. B is known to be 2500′ south and 600′ west of C. Find the length and bearing of the straight passage from A to B.

7. A horizontal line 1033 ft. is measured in the same vertical plane with the top of a mountain. From one end of the line the angle of elevation of the top of the mountain is 13° 22′, from the other end the angle of elevation is 5° 10′. What is the height of the mountain above the horizontal line?

8. On the right bank of a river, two stakes A and B are set 250.0′ apart on a horizontal line. On the left bank a stake C is set so that the angle A in the triangle ABC is 90°. The angle B is measured to be 38° 41′. What is the distance from A to C?

9. From a hill top the angles of two points on opposite sides of the hill are 20° 33′ and 15° 10′, respectively. The distance from the hill top to the first point is 1200 yds. The distance to the second point 2000 yds Find the distance between the points.

10. A surveyor took measurements as follows: $AB = 500'$, angle $DAB = 100°$, angle $DAC = 67° 45'$, angle $ABC = 125°$, angle $DBC = 70°$. Find the length of DC.

11. A flag pole stands on a tower 50′ high. From A, on a level with the base of the tower, the angle of elevation of the top of the tower is 42° 35′. From A the pole subtends an angle of 10° 15′. Find the length of the flag pole.

12. Find the difference between the areas of the triangle, ABC, and its inscribed and circumscribed circles, if the sides of the triangle are $a = 56.3$, $b = 76.5$, $c = 68.8$.

13. What are the angles of the triangle whose sides are, $a = 665$, $b = 776$, $c = 887$?

14. Two stones are one mile apart on a straight road. From a tower standing on the roadside the angles of depression of the stones are 15° and 3° 50′, respectively. Find the height of the tower and its distance from each stone.

Note. — There are two cases: (a) when the tower is between the stones; (b) when the stones are on the same side of the tower.

70. Graphic representation of the trigonometric functions.— (a) Construct the graph of the equation

$$y = \sin x.$$

To do this, the values of the angle x must be laid off in linear units to some scale. From **68**, if $r = 1$, the arc = the angle at the center expressed in radians. This furnishes the unit of

measure, *viz.*, an arc equal in length to the radius. Lay off values of x along the x-axis to the scale of 1 radian = 1″.

Tabulate values of x, and $y = \sin x$ below.

x	$y = \sin x$.	x	$y = \sin x$.
$0° = 0$ radians* . . .	0.0	$210° = 7\pi/6$ radians . . .	−0.5
$30° = \pi/6$ " . . .	0.5	$225° = 5\pi/4$ " . . .	−0.707
$45° = \pi/4$ " . . .	0.707	$240° = 4\pi/3$ " . . .	−0.866
$60° = \pi/3$ " . . .	0.866	$270° = 3\pi/2$ " . . .	−1.0
$90° = \pi/2$ " . . .	1.0	$300° = 5\pi/3$ " . . .	−0.866
$120° = 2\pi/3$ " . . .	0.866	$315° = 7\pi/4$ " . . .	−0.707
$135° = 3\pi/4$ " . . .	0.707	$330° = 11\pi/6$ " . . .	−0.5
$150° = 5\pi/6$ " . . .	0.5	$360° = 2\pi$ " . . .	0.0
$180° = \pi$ " . . .	0.0		

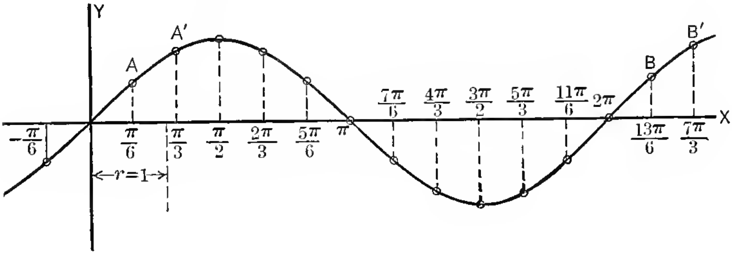


FIG. 52.

A property of trigonometric functions known as periodicity is illustrated by this curve. This was suggested in **52.** It is noticed that as a point traces the above curve the values of y are the same for positions of the point going in the same direction on the curve, such as the pairs of points A, B; A', B'; A'', B''. It follows that corresponding to any value of $\sin x$, there are infinitely many values of x differing successively by 360° or 2π radians. This fact may be expressed by

$$\sin x = \sin (x \pm n\,360°) = \sin (x \pm 2\,n\pi),$$

* Note that $\pi = 3.1416$, then $\pi/6$ radians $= 0.5236$ radians and similarly with other values.

where $n = 0, 1, 2, 3, \ldots$ A similar relation holds for each of the functions $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$.

It may be further noted that there are two equal ordinates of the curve $y = \sin x$ in each of the intervals 0° to 180°; 180° to 360°; 360° to 540°, etc. From this fact it occurs that the value of a single function alone, say of $\sin x$ or $\tan x$, is not sufficient to determine which of the two values of x between 0° and 180° or between 180° and 360° is to be taken. Either another function must be known or else the quadrant in which the angle lies must be known. This fact was illustrated in the solution of the ambiguous case of triangles, **63.**

1. Construct the graph of $y = \cos x$ in a manner similar to that employed in constructing the graph of $y = \sin x$, above.

2. Construct the graph of $y = \sin x + \cos x$.

Note. — Tabulate values of $\sin x$ and $\cos x$ and add the corresponding values. Use these sums as ordinates of the curve.

3. Construct the graph of $y = \sin 2x$.

Note. — Erect the ordinates equal to $\sin 2x$, over the values of x.

4. In a manner similar to that indicated in Ex. (3), construct the graph of $y = \sin 3x$. How do the curves in this exercise and the preceding differ from the graph of $y = \sin x$?

5. Construct the graph of $y = \sin \frac{1}{2} x$.

6. Construct the graph of $y = \tan x$. What peculiarity has this curve at $x = \pi/2$, $x = 3\pi/2$, etc.?

7. Construct the graph of $y = \dfrac{1}{x} \sin x$, given that the value of $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right) = 1$.

8. Construct the graph of $y = \sec x$. What peculiarity does this curve have at $\pi/2$, $3\pi/2$, etc.? In what way does it differ from the sine curve and the tangent curve at $x = 0$, $x = \pi$, etc.?

71. The equation $y = \text{arc} \sin x$ may also be studied by means of its graph. This function is infinitely many valued as was shown in **52.**

Values of $y = \text{arc} \sin x$ for different values of x.

0	$0°$	$\pm 360°, \pm 720°, \ldots$; $\pm 2\pi, \pm 4\pi, \ldots$
0.5	$30°$, $\frac{\pi}{6}$,	$30° \pm 360°, \ldots$; $\frac{\pi}{6} \pm 2\pi, \ldots$
	$150°$, $\frac{5\pi}{6}$,	$150° \pm 360°, \ldots$; $\frac{5\pi}{6} \pm 2\pi, \ldots$
0.866	$60°$, $\frac{\pi}{3}$,	$60° \pm 360°, \ldots$; $\frac{\pi}{3} \pm 2\pi, \ldots$
	$120°$, $\frac{2\pi}{3}$,	$120° \pm 360°, \ldots$; $\frac{2\pi}{3} \pm 2\pi, \ldots$
-1.0	$90°$, $\frac{\pi^2}{2}$,	$90° \pm 360°, \ldots$; $\frac{\pi}{2} \pm 2\pi, \ldots$
-0.5	$210°$, $\frac{7\pi}{6}$,	$210° \pm 360°, \ldots$; $\frac{7\pi}{6} \pm 2\pi, \ldots$
	$330°$, $\frac{11\pi}{6}$,	$330° \pm 360°, \ldots$; $\frac{11\pi}{6} \pm 2\pi, \ldots$
-0.866	$240°$, $\frac{4\pi}{3}$,	$240° \pm 360°, \ldots$; $\frac{4\pi}{3} \pm 2\pi, \ldots$
	$300°$, $\frac{5\pi}{3}$,	$300° \pm 360°, \ldots$; $\frac{5\pi}{3} \pm 2\pi, \ldots$
-1.0	$270°$, $\frac{3\pi}{2}$,	$270° \pm 360°, \ldots$; $\frac{3\pi}{2} \pm 2\pi, \ldots$

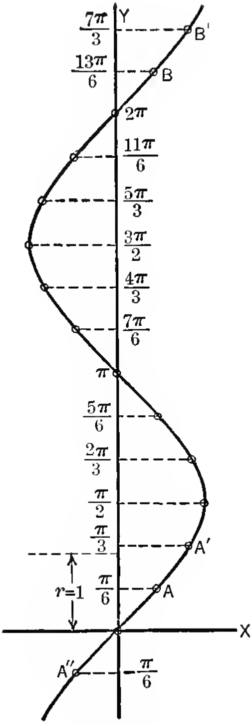


FIG. 53.

The figure shows the graph for values of y from $-\frac{\pi}{6}$ to $+\frac{7\pi}{3}$.

The curve shows that any line parallel to the y-axis and less than a unit distant from it cuts the curve in many points, that is, there are many ordinates for each abscissa. Such a function as $y = \text{arc} \sin x$ is called a multivalued function of x. The smallest positive ordinate for any value of the abscissa is called the principal value of the ordinate or angle.

1. Construct the graph of $y = \text{arc} \cos x$.
2. Construct the graph of $y = \text{arc} \tan x$.
3. Construct the graph of $y = \text{arc} \sin 2x$.
4. Construct the graph of $y = \text{arc} \sin \frac{1}{2} x$.

72. Equations involving **trigonometric functions** as unknowns are of frequent occurrence. They may be solved in much the same way as algebraic equations.

1. Find the principal value of x that will satisfy the equation

$$2 \sin^2 x - 3 \cos x = 0.$$

This can be written,

$$2(1 - \cos^2 x) - 3 \cos x = 0.$$

Solving for $\cos x$,

$$\cos x = 0.5 \quad \text{or} \quad 2.$$

$$\therefore \quad x = \text{arc} \cos 0.5 \quad \text{or} \quad \text{arc} \cos 2.$$

The first gives $x = 60°$ and $300°$, of which $60°$ is the principal value. The second is impossible since $|\cos x| \leqq 1$. This value must, therefore, be rejected.

2. Solve and determine the principal value of x in

$$2 \sin 2x - \cos^2 x = 0.$$

Note. — First remove the double angle by substituting its equivalent from formulas of **53**.

3. Solve $\cos^2 \theta - \sin \theta = \frac{1}{4}$.
4. " $4 \cos \theta = 3 \sec x$.
5. " $3 \sin \theta - 2 \cos^2 \theta = 0$.
6. " $\tan \theta + \cot \theta = 2$.
7. " $4 \sec \theta - 7 \tan^2 \theta = 3$.

Note. — It may happen in solving an equation containing two functions, that eliminating one function will lead to only part of the solution. The other part of the solution may be obtained by eliminating the other function.

72*a*. — Tables requiring **special interpolation.** — Gradients are commonly called grades or slopes and are expressed:

(*a*) By the angle which the line of direction makes with a horizontal line.

Example. — A gradient of 1°, 2°, 3°, etc.

(*b*) By the change of elevation corresponding to a given horizontal distance.

Example. — An elevation of 4′ in 100′, that is, a 4 per cent slope.

An elevation 1′ in 60′ gives a gradient of $\frac{1}{60}$ (read: 1 on 60).

Below is a table which gives differences of elevation for gradients of 0° to 5°, and horizontal distances.

Gradient in degrees.	Difference of elevation for horizontal distance of:								
	1	2	3	4	5	6	7	8	9
0.5	0.0087	0.0174	0.0261	0.0348	0.0435	0.0522	0.0609	0.0696	0.0783
1	0.0175	0.0350	0.0525	0.0700	0.0875	0.1050	0.1225	0.1400	0.1575
2	0.0349	0.0698	0.1047	0.1396	0.1745	0.2094	0.2443	0.2792	0.3141
3	0.0524	0.1048	0.1572	0.2096	0.2620	0.3144	0.3668	0.4192	0.4716
4	0.0699	0.1398	0.2097	0.2796	0.3495	0.4194	0.4893	0.5592	0.6291
5	0.0875	0.1750	0.2625	0.3500	0.4375	0.5250	0.6125	0.7000	0.7875

Find the difference of elevation for a gradient of 3°, and a horizontal distance of 567 ft.

From table, on line with gradient of 3°,

For distance 7 the elevation is	0.3668
For distance 60 = 6 × 10 elevation is.	3.1440
For distance 500 = 5 × 100 elevation is. . . .	26.2000
Total.	29.7108

For distance 567 ft., therefore, elevation is 29.71 ft.

EXERCISES.

1. Check the above method of interpolation by using your table of logarithms and the formula: elevation = horizontal distance $\times$ tangent of the angle.

2. Make out a table corresponding to the table above replacing horizontal distances by distances measured along the slope, using the formula: elevation = distance along slope $\times$ sine of angle.

3. Compare this table with one in text. What conclusion do you draw when the angle is small?

In solving problems connected with traverse sailing or with land surveying, it is often necessary to find the latitude and departure corresponding to the several courses and distances.

Since *latitude* = *distance* $\times$ *cos of bearing*,
departure = *distance* $\times$ *sin of bearing*,

it is convenient to use a table in which these projections have been computed. Following is a section of such a table:

TRAVERSE TABLE.

Course.	Dist. 1.		Dist. 2.		Dist. 3.		Dist. 4.		Dist. 5.		Course.
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	
30° 15′	0 8638	0.5038	1.7277	1.0075	2.5915	1.5113	3 4553	2.0151	4.3192	2.5189	59° 45′
30	8616	5075	7233	0151	5849	6226	4465	0302	3081	5377	30
45	8594	5113	7188	0226	5782	5339	4376	0452	2970	5565	15
31 0	8572	5150	7142	0301	5716	5451	4287	0602	2858	5752	59 0
15	8549	5188	7098	0375	5647	5563	4196	0751	2746	5939	45
30	8526	5225	7053	0450	5579	5675	4106	0900	2632	6125	30
45	8504	5262	7007	0524	5511	5786	4014	1049	2518	6311	15
32 0	8480	5299	6961	0598	5441	5898	3922	1197	2402	6496	58 0
15	8457	5336	6915	0672	5372	6008	3829	1345	2286	6681	45
30	8434	5373	6868	0746	5302	6119	3736	1492	2170	6865	30
45	0 8410	0.5410	1 6821	1.0810	2.5231	1.6229	3.3642	2 1639	4.2052	2.7049	15
33 0	8387	5446	6773	0893	6160	6339	3547	1786	1934	7232	57 0
Course.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Course.
	Dist. 1.		Dist. 2.		Dist. 3.		Dist. 4.		Dist. 5.		

From this table determine the latitude and departure of a course with bearing 31° 10′ and length (distance) 413.

Determine the latitude and departure of a course with bearing 31° 10′ and length 413.

A surveyor runs a line N. 30° 45′ E., distance 243, then N. 32° 15′ W., distance 214. How far north and how far east or west is he from the starting point?

By use of the traverse table, find the area of a right triangle which has a hypotenuse 534 units in length and an acute angle 59° 10′.

A table of logarithmic and natural haversines is indispensable in connection with problems in nautical astronomy. Below is a portion of such a table.

LOGARITHMIC AND NATURAL HAVERSINES

$$\text{hav}\ \theta = \tfrac{1}{2}\ \text{vers}\ \theta = \tfrac{1}{2}\,(1 - \cos\theta) = \sin^2\theta/2.$$

s.	$3^h\ 50^m$ 57° 30′.		$3^h\ 51^m$ 57° 45′.		$3^h\ 52^m$ 58° 0′.		$3^h\ 53^m$ 58° 15′.		$3^h\ 54^m$ 58° 30′.		s.
	Log hav.	Nat. hav.	Log hav.	Nat. hav.	Log hav.	Nat. hav.	Log. hav.	Nat. hav.	Log. hav.	Nat. hav.	
0	9.36427	.23135	9.36772	.23319	9.37114	.23504	9.37455	.23689	9.37794	.23875	60
1	.36433	.23138	.36777	.23322	.37120	.23507	.37461	.23692	.37800	.23878	59
2	.36439	.23141	.36783	.23325	.37126	.23510	.37467	.23695	.37806	.23881	58
3	.36444	.23144	.36789	.23329	.37131	.23513	.37472	.23699	.37811	.23884	57
+1′	9.36450	.23147	9.36794	.23332	9.37137	.23516	9.37478	.23702	9.37817	.23887	56
5	.36456	.23150	.36800	.23335	.37143	.23519	.37484	.23705	.37823	.23891	55
6	.36462	.23153	.36806	.23388	.37148	.23523	.37489	.23708	.37828	.23894	54
7	.36467	.23156	.36812	.23341	.37154	.23526	.37495	.23711	.37834	.23897	53
+2′	9.36473	.23160	9.36817	.23344	9.37160	.23529	9.37501	.23714	9.37840	.23900	52
	$20^h\ 9^m$		$20^h\ 8^m$		$20^h\ 7^m$		$20^h\ 6^m$		$20^h\ 5^m$		

Solve the following problems by use of above table (do not determine θ).

Log hav θ = 9.37144–10, find nat. hav θ.
Nat. hav θ = 0.23893, find log hav θ.
t (time) = $3^h\ 51^m\ 3^s$, find log hav t and nat. hav t.
t = $20^h\ 7^m\ 56^s$, find log hav t and nat. hav t.
Find log hav and nat. hav of 58° 0′ 25″.

MISCELLANEOUS EXERCISES

1. The angle at the center of a circle is 4 radians. The arc is 7′. What is the radius of the circle?

2. What is the value, in degrees of $\frac{5\pi}{7}$ radians? Of $\frac{2\pi}{5}$ radians?

3. What is the value in radians of 40°? 55°? 1200°?

4. An arc is 5′. The radius is 8′. What is the angle at the center in radians? In degrees?

5. One angle exceeds another by $\frac{2\pi}{5}$ radians. The sum of the angles is 175°. Find the angles.

6. Prove $\sin a \cos a = \sin^3 a \cos a + \cos^3 a \sin a$.

7. Prove $\cot a \csc a = 1/(\sec a - \cos a)$.

8. Prove $(1 + \tan^2 a)/(1 + \cot^2 a) = \sin^2 a/\cos^2 a$.

9. Prove $\sin a - \sin a/(\cot a - 1) = \sin \alpha\,[1 - \sin \alpha/(\cos \alpha - \sin \alpha)]$.

10. Given $\tan 30° = \frac{1}{3}\sqrt{3}$, find tan 15°, by use of a formula.

11. Using the addition theorem find $\sin(a + b + c)$.

12. Prove $\cos 2a = (1 - \tan^2 a)/(1 + \tan^2 a)$.

13. Prove $\sin\left(\frac{\pi}{4} - a\right) = (\cos a - \sin a)/\sqrt{2}$.

14. Prove $\tan a + \tan b = \sin(a + b)/\cos a \cos b$.

15. Prove $\cos a = (1 - \tan^2 \frac{1}{2} a)/(1 + \tan^2 \frac{1}{2} a)$.

16. Prove $\cos(45° + a)/\cos(45° - a) = \sec 2a - \tan 2a$.

Solve and determine all values less than 360°.

17. $\tan a = \sin a$.

18. $(3 - 4\cos^2 a)\cos 2a = 0$.

19. $\sin a + \cos a \cot a = 2$.

20. $\tan(45° + a) = 3\tan(45° - a)$.

21. $5 \sin a = \tan a$.

22. $2\cos a + \sec a = 3$.

23. $2\sin a + 5\cos a = 2$.

24. $\tan 2a = 1$.

25. If $\tan 2a = m$, find $\tan a$.

26. If $\cos a = \frac{1}{2}$, find $\cos \frac{1}{2} a$.

27. Prove $\text{arc tan } \frac{1}{4} + \text{arc tan } \frac{3}{5} = \frac{\pi}{4}$, given $\tan \frac{\pi}{4} = 1$.

28. Solve for y in $2 \text{ arc sin } \frac{1}{2} + \text{arc sin } y = \pi/2$.

29. In the figure the following data are given to find AB.

$CD = 943.4$; $CE = 673.3$; $\alpha = 72° \, 9.3′$; $\beta = 60° \, 17.9′$; $\gamma = 32° \, 14.6′$; $\delta = 67° \, 33.9′$; $\epsilon = 19° \, 14.7′$ See Fig. 54.

Ans. $AB = 1054$.

30. A coal mine entry runs N. 18° W. to A whose coördinates are (450, 100). A post on the boundary line has coördinates (575, 475). The bearing of the

boundary line is N. 26° 50′ E. Another entry has been run N. 80° E. to a point B across the boundary whose coördinates are (112, 174). It is desired to continue the first entry to a point 10′ from the boundary, then parallel to the boundary to a point where the second entry would intersect it if continued,

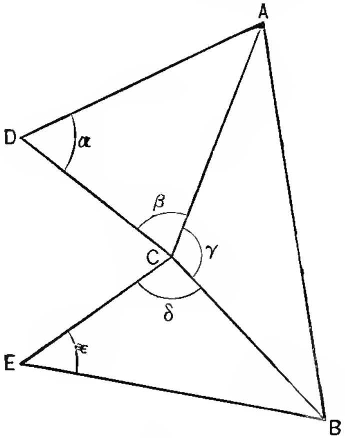


FIG. 54.

then to follow a line that would coincide with the continuation of the second entry. Make the calculations and determine the coördinates of the points of turning.

31. Find the coördinates of P from the following: Start at A whose coördinates are $(-75, 350)$; run S. 26° 15′ W., 355′; then run S. 54° 20′ E., 175′; then run N. 10° 15′ E., 300′; then run N. 75° 45′ W., 500′ to P.

32. Since the mil is the angle at the center of a circle subtended by $\frac{1}{6400}$ of the circumference, therefore,

$$\frac{\text{Measure in mils of angle } A}{\text{Measure in degrees of angle } A} = \frac{6400}{360} = \frac{160}{9}.$$

Change the following angles in degrees to mils: 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°.

By means of the definition of a mil and that of a radian determine the conversion factor for changing angles from mils to radians. Change the following angles in radians to mils: $\pi/2$, $\pi/4$, π.

Change the following angles in mils to radians: 800, 1600, 2400, 3200.

33. By the definition of a mil, if the radius of a circle is 1000 yds., an arc of 1 yd. subtends an angle at the center of approximately one mil. In

the figure, therefore, from similar triangles, assuming f and arc s equal,

$$\frac{s}{F} = \frac{1000}{R} \quad \text{or} \quad s = \frac{1000}{R} \cdot F,$$

or, since the angle t in mils has the same measure as the arc s,

$$t = \frac{1000}{R} \cdot F.$$

Note. — The assumptions made in the above discussion give results sufficiently accurate, in general, for work in gunnery where t is usually a very small angle.

(*a*) If the range $R = 1000$ yds., and the distance between the two guns $F = 20$ yds., what is t in mils?

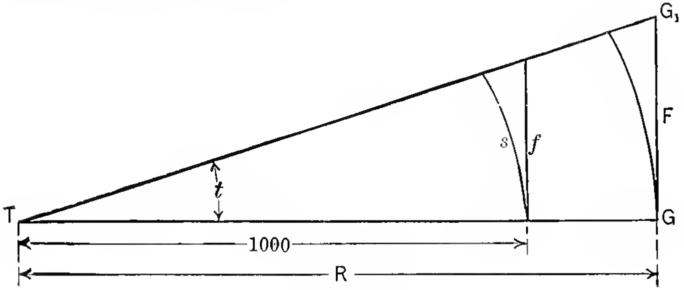


FIG. 55.

What is t if

(*b*) $R = 1000$ yds., and $F = 40$ yds.?
(*c*) $R = 1000$ yds., and $F = n$ yds.?
(*d*) $R = 2000$ yds., and $F = 20$ yds.?
(*e*) $R = 2500$ yds., and $F = 30$ yds.?

34. The angle at the target T (or aiming point P) subtended by the distance between two successive pieces G and G_1 on the battery front is called the parallax of the target (or aiming point). If the aiming point is in the line of the battery front the parallax of the point is a minimum, zero. If the target (aiming point) is on the normal to the battery front, GG_1 at the mid-point the parallax is a maximum. This is called the *normal parallax*. In practice, if the target is on any normal to the battery front, its parallax is considered as normal. In what follows the normal through G is chosen.

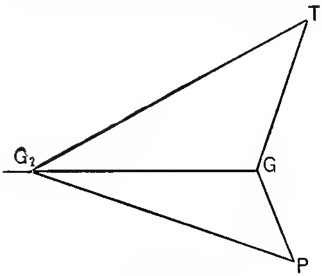


FIG. 56.

The normal parallax t of T is (see exercise 33, above)

$$t = \frac{GG_1}{R/1000} = \frac{F}{R/1000},$$

where R is the distance from T to battery front.

(a) What is the normal parallax of the target for a range 2500 yds., and distance 20 yds. between G and G_1?

(b) What is the normal parallax if R = 500 yds. and F = 40 yds.?

35. If the target is not on the normal to the battery front, a correction for "obliquity" is required. This is called "correction of parallax due to obliquity."

The angle at G between the line normal to the battery front and the line to the target T' is the angle of obliquity. In the figure, O is the angle of obliquity of the target T'.

The **true parallax** (t) is the normal parallax after it is corrected for obliquity. True parallax is obtained as follows:

$$DG = G_1G \cos DGG_1, \text{ approximately,}$$
$$= G_1G \cos O.$$

But

$$t' \text{ (mils)} = DG \frac{1000}{R} \qquad \text{(See Exercise 33.)}$$

Therefore

$$t' \text{ (mils)} = G_1G \frac{1000}{R} \cos 0$$
$$= t \text{ (mils)} \cos 0,$$

since

$$G_1G \frac{1000}{R} = \text{normal parallax of } T.$$

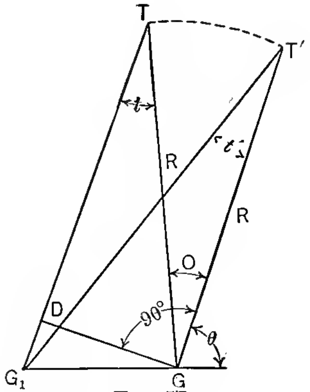


FIG. 57.

Therefore, true parallax = normal parallax × cosine of the angle of obliquity.

(a) If the normal parallax t = 8 mils, and the obliquity angle is 600 mils, find the true parallax.

(b) If R = 2500 yds., the obliquity angle 600 mils, and G_1G = 20 yds., find the true parallax.

(c) Verify the formula, true parallax $t' = \dfrac{GG_1 \sin \theta}{R/1000}$, where angle θ is as indicated in the figure.

CHAPTER IX

POLAR COÖRDINATES, COMPLEX NUMBERS, VECTORS

73. The **position** of a point can be determined by means of a **distance** and an **angle**. Let OX be a fixed reference line, called the axis. Call O the pole. Then if the angle θ, and the distance $OP = r$, are known the position of P is known. The point P is designated as $P\ (r, \theta)$ or as (r, θ). The number pair (r, θ) are the **polar** coördinates of P. $OP = r$ is the **radius vector** of P, and θ is the **vectorial angle** of P.

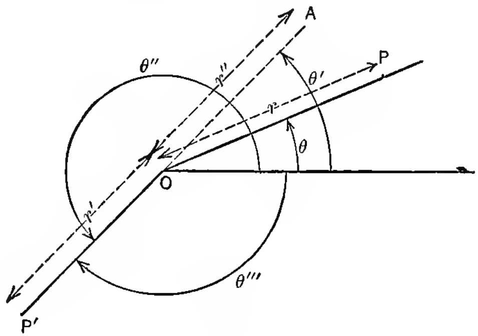


FIG. 58.

If P lies on the terminal line of the vectorial angle, r is reckoned as positive. If P lies on the terminal line of the vectorial angle produced on the opposite side of the pole, r is regarded as negative. The vectorial angle θ is positive or negative according as it is reckoned counter clockwise or clockwise, respectively. In the figure, if P' is regarded as $P'\ (r'\theta'')$, r' and θ'' are positive. If P' is regarded as (r', θ'), r' is negative. If P' is regarded as

$(r'\theta''')$, r' is positive and θ''' is negative, and similarly in other cases.

1. Locate the following points $\left(4, \frac{\pi}{4}\right)$, $(6, 30°)$, $\left(-5, \frac{\pi}{3}\right)$, $\left(-2, -\frac{2\pi}{3}\right)$.

2. Construct graph of $r = 8\,\theta$ (θ in radians).

Note. — Arrange a table of values as in drawing graphs on rectangular coördinates. Then lay off the points as above and draw a smooth curve through the points.

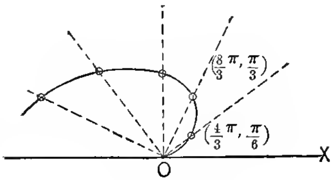


FIG. 59.

Assume $\theta = 0, \quad \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \cdots$

Calculate $r = 0, \; 8 \cdot \frac{\pi}{6}, \; 8 \cdot \frac{\pi}{3}, \; 8 \cdot \frac{\pi}{2} \cdot \cdots$

A portion of the graph is shown in the figure.

3. Construct the graph of $r = 8 \cos \theta$.
4. " " $r = 2/(1 - \cos \theta)$.
5. " " $r = \sin 2\,\theta$.
6. " " $r \sin \theta = 4$ (solve for r).
7. " " $r = 4/(1 - 3 \cos \theta)$.
8. " " $r = 7$ (a circle).
9. " " $r = \cos 3\,\theta$.
10. " " $r = \cos \frac{\theta}{2}$.

11. In $2x + 3y = 5$, put $x = r \cos \theta$, $y = r \sin \theta$ and draw the graph of the resulting equation. Draw the graph from the original equation.

12. In $y^2 = 8x$, put $y = r \sin \theta$, $x = r \cos \theta + 2$. Note the form of the resulting equation. Construct the graphs of both equations.

74. In the solution of certain quadratic equations and equations of higher degree, there occur roots of the form $a + bi$, where a, b are real and $i^2 = -1$ or $i = \sqrt{-1}$, **36**g.

From the above definition of i it is easily found by calculating that,

$$\begin{aligned}
i &= \sqrt{-1},\\
i^2 &= (\sqrt{-1})^2 = -1,\\
i^3 &= i^2 \cdot i = (-1)\sqrt{-1} = -\sqrt{-1} = -i,\\
i^4 &= i^3 \cdot i = -i \cdot i = -(i)^2 = 1,\\
i^5 &= i^4 \cdot i = 1 \cdot i = i.
\end{aligned}$$

The value of i^5 is the value of i, the value of i^6 will be the value of i^2, etc.

75. A **geometric basis** for interpreting complex numbers is attributed to Argand. Since any number or any line segment has its sign or direction reversed when multiplied by $-1 = i^2$, the line is rotated 180° counter clockwise by multiplying by $-1 = i^2$. Thus in the figure $AB \cdot (-1) = AB \cdot i^2 = -AB = AB'$. It then looked reasonable to suppose that AB is rotated 90° counter clockwise when multiplied by $\sqrt{-1} = i$ or AB is just half reversed. Thus $AB \cdot i = AB''$ in the figure. This idea proves to be very useful. The number $a + bi$ may now be looked upon as a point in the plane determined by two steps taken perpendicular to each other. Thus, if $a = Om$ and $b = mn$ in Fig. 61, $a + bi$ will be $Om + mP$, since bi is perpendicular to b. This idea is then equivalent to regarding (a, b) in $a + bi$ as the rectangular coördinates of a point P.

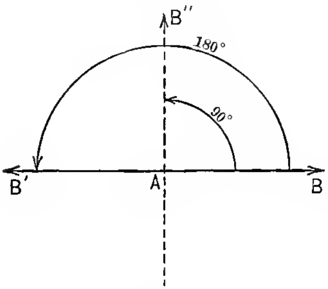


FIG. 60.

Polar coördinates may also be associated with P. Thus, if angle $XOP = \theta$, and $OP = r$, P may be regarded as $P\ (r, \theta)$.

The line $OP = r$ is called the **modulus** or absolute value of

the number $a + bi$. The angle θ is called its **amplitude**. From the figure it is easy to see that for any position of P,

(1) $$r = \sqrt{a^2 + b^2}.$$

(2) $$\sin\theta = \frac{b}{r}.$$

(3) $$\cos\theta = \frac{a}{r}.$$

(4) $$\tan\theta = \frac{b}{a}.$$

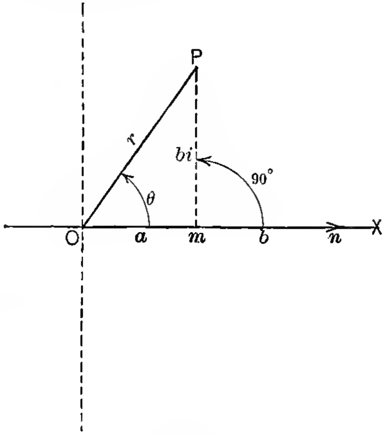


FIG. 61.

These relations are fundamental.

1. Locate on rectangular coördinate paper the points, $2 + 3i$ ($a = 2$, $b = 3$); $2 - 3i$; $1 - 4i$; $-4 - 3i$; $-5 + 2i$.

2. Find the modulus and amplitude of each number in (1).

76. Arithmetic operations with complex numbers. —

(*a*) Two complex numbers are **equal** when and only when they represent the **same point** referred to the same axes. Hence the two complex numbers $a + bi$ and $a' + b'i$ are equal when and only when $a = a'$ and $b = b'$.

(*b*) The **sum** of two complex numbers $a + bi$ and $a' + b'i$ is the complex number $a + a' + (b + b')i$. Thus the sum of $3 + 2i$ and $1 + 4i$ is $4 + 6i$.

1. Add $3 + 5i$ to $1 + 2i$; $1 + 6i$ to $3 - 2i$; $-\frac{1}{2} - \frac{1}{2}i$ to $-\frac{1}{2} + \frac{1}{2}i$, locate each point and each sum on a diagram in rectangular coördinates.

2. Add $(-2 + 3i)$, $(-1 - 2i)$, $(3 + 6i)$. Locate all points and the sum.

3. Add $(1 + 3i)$, $(-3 - i)$, $(6 + 7i)$. Locate all points and the sum.

(*c*) **Multiplication** of two complex numbers is defined by

$$(a + bi)(a' + b'i') = aa' - bb' + (ab' + a'b)i.$$

Note. — Remember in carrying out the work, $i^2 = -1$.

(*d*) **Division** of complex numbers is defined by

$$\frac{a+bi}{a'+b'i}=\frac{(a+bi)\,(a'-b'i)}{(a'+b'i)\,(a'-b'i)}=\frac{aa'+bb'}{a'^2+b'^2}+\frac{(a'b-ab')\,i}{a'^2+b'^2}.$$

The operations (*b*), (*c*), (*d*) are in general possible and lead to complex numbers, impossible if $a'^2+b'^2=0$, indeterminate if $a'^2+b'^2=a^2+b^2=0$, see **42, 43, 77.**

1. $(3-4\,i)\,(7+4\,i)=?$ $(3-4\,i)\,(-7-4\,i)=?$ Locate all points and the products.

2. $(3+4\,i)\div(7+4\,i)=?$ $(3-4\,i)\div(-7-4\,i)=?$ Locate all points and the quotients.

3. $(2+3\,i)\div(1-i)=?$ $(1-i)$ $(1+i)\div 2+2\,i=?$ Locate all points and the quotients.

77. By the formulas of **73**, $a+bi$ may be put in the form

$$a+bi=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+\frac{bi}{\sqrt{a^2+b^2}}\right)$$
$$=r\,(\cos\theta+i\sin\theta).$$

The last form is called the polar form of the complex number $(a+bi)$, where

$$r=\sqrt{a^2+b^2},\ \sin\theta=\frac{b}{\sqrt{a^2+b^2}}\ \text{ and }\ \cos\theta=\frac{a}{\sqrt{a^2+b^2}}.$$

Multiplication and division of complex numbers take very interesting and useful forms in polar coördinates. Thus

$$(a+bi)\ (a'+b'i)=r\,(\cos\theta+i\sin\theta)\,r'\,(\cos\theta'+i\sin\theta')$$
$$=rr'\,[(\cos\theta\cos\theta'-\sin\theta\sin\theta')+i\,(\sin\theta\cos\theta'+\cos\theta\sin\theta')]$$
$$=rr'\,[\cos\,(\theta+\theta')+i\sin\,(\theta+\theta')].$$

This shows that the modulus of the product is the product of the moduli and the amplitude of the product is the sum of the amplitudes of the numbers multiplied.

Again,

$$\frac{a+bi'}{a'+b'i}=\frac{r\,(\cos\theta+i\sin\theta)\,(\cos\theta'-i\sin\theta')}{r'\,(\cos\theta'+i\sin\theta')\,(\cos\theta'-i\sin\theta')}$$
$$=\frac{r}{r'}\,[(\cos\,(\theta-\theta')+i\sin\,(\theta-\theta')].$$

This shows that the modulus of the quotient is the modulus of

the dividend divided by the modulus of the divisor and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.

As an exercise reduce each complex number in the last section to polar form and perform the indicated operations by use of the above formulas.

78. There are a number of physical quantities, of fundamental importance, such as force, velocity, electric field, etc., that are completely specified by **magnitude** and **direction** of the line of action. They are called **vector quantities** in distinction from non-directed quantities, such as energy, speed, etc. The latter are called ***scalar*** quantities for the reason that they are completely specified by magnitude alone.

A directed straight line segment is a **vector.** It is evident that a vector may represent a vector quantity. For the length of the vector, to some scale, may represent the magnitude of the vector quantity and the vector may be parallel to the line of action of the vector quantity.

If AB is a vector, its direction is understood to be from A toward B. A is the initial point of the vector AB, and B is the terminal point.

A vector may be displaced parallel to itself without affecting its magnitude or direction. When a vector may be so moved it is called a free vector. When a vector is attached to a fixed point, it is a fixed or localized vector.

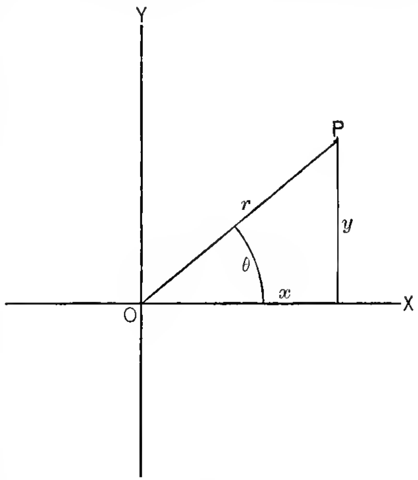


FIG. 62.

79. The notation of complex numbers and of polar coördinates lends itself readily to representing vectors and to cal-

culating with them. Thus the vector OP in Fig. 62 is described either as $(x + iy)$ or as (r, θ) at pleasure. Since a vector symbolizes a vector quantity, any operations of arithmetic performed with vectors will have a similar meaning with vector quantities. The direction of OP is that from O to P. When OX is the reference line (axis) the angle θ determines the direction of OP.

80. Addition and subtraction of vectors. — The most convenient notion from which to derive vector addition and subtraction is displacement or step. Thus to add a vector $BC = \boldsymbol{b}$* to another vector $AB = \boldsymbol{a}$, lay off $\boldsymbol{a}$ and then from the

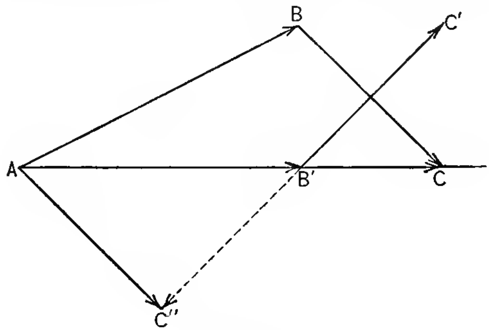


FIG. 63.

terminal point of $\boldsymbol{a}$ lay off $\boldsymbol{b}$. The vector from the initial point of $\boldsymbol{a}$ to the terminal point of $\boldsymbol{b}$, that is, $AC = \boldsymbol{c}$, is the vector sum of $\boldsymbol{a}$ and $\boldsymbol{b}$. This may be expressed by the vector equation

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}.$$

That is, a displacement $AB = \boldsymbol{a}$ followed by the displacement $BC = \boldsymbol{b}$ is equivalent to the displacement $AC = \boldsymbol{c}$.

If the vector $B'C'$ is to be subtracted from the vector AB', lay off the vector $B'C''$ equal in length to $B'C'$ but in the opposite direction. The vector AC'' is the difference. If $AB' = \boldsymbol{a}'$, $B'C' = \boldsymbol{b}'$ and $AC'' = \boldsymbol{c}'$ we may write the vector equation

$$\boldsymbol{a}' - \boldsymbol{b}' = \boldsymbol{c}'.$$

* When a single letter represents a vector heavy type is used.

It is seen that to subtract a vector is the same as to add its opposite or negative.

To add several vectors OP, OP', OP'', . . . lay off the vectors successively, with initial point of each on the terminal point of the preceding, forming the sides of a polygon. The closing side of the polygon drawn from the starting point to the terminal point of the last vector is the vector sum of the given vectors, Fig. 65. The vector equation is

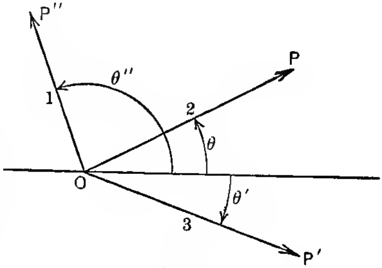


FIG. 64.

$$OP'' + OP + OP' = O'P'.$$

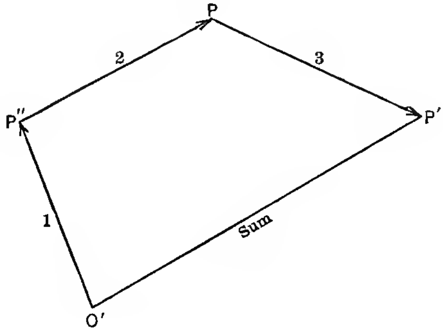


FIG. 65.

It must be remembered we are not adding the lengths of the lines alone but the displacements (including directions of the lines). The sum of two sides of a triangle is **vectorially** equal to the third side, but not **numerically**.

1. Find the vector sum of two adjacent sides of a parallelogram $ABCD$. Find the vector difference of the same sides.

2. Find the vector sum of the three sides of a triangle ABC,

taken in order. Find the vector difference $AB - BC$. Find the vector sum $AB + BC$.

3. Find the sum of the vectors whose initial points are at the pole and whose terminal points are $(3, 60°)$; $(10, -20°)$; $(-24, 120°)$, respectively.

4. Find the sum of the vectors represented by $(3 - 6i)$; $(4 + 2i)$; $(5 - 3i)$; $(1 - 4i)$.

5. Find the magnitude and direction of the velocity resulting from two simultaneous velocities, one due north 50′/sec., the other N. 45° E., 30′/sec.

Note. — By **75, 76** these velocities may be expressed in rectangular form and added. The sum can then be reduced to polar coördinates. The result may be found by use of the law of **58.** Solve by both methods and check the results.

6. A boat is rowed across the current of a river at 5 mi./hr. The current is $1\frac{1}{2}$ mi./hr. Find the actual velocity of the boat in magnitude and direction.

7. If a horse is running N. 35° E. at the rate of 12 mi./hr., how fast is he going north? How fast east?

8. Make diagrams to scale for Exs. 6, 7 and verify your calculations.

9. Can you explain by the vector idea why a division of society into opposing factions retards or prevents social progress?

10. Show by vector addition the truth of the parallelogram of forces.

81.* Product of Two Vectors. — Vectors exhibit two types of product:

(*a*) **Scalar product** of two vectors is defined as the product of their magnitudes and the cosine of their included angle. Thus if

$$\boldsymbol{F} = r(\cos\theta + i\sin\theta)\ (\mathbf{77, 79})$$

and

$$\boldsymbol{S} = r'(\cos\theta' + i\sin\theta'),$$

* The remainder of this chapter may be omitted if desired. It is recommended, however, if the time permits and the student's knowledge of mechanical notions justifies, that the entire chapter be covered carefully.

the scalar product of $\boldsymbol{F}$ and $\boldsymbol{S}$ will be written as

$$\boldsymbol{FS} = rr' \cos(\theta' - \theta) = FS \cos(\theta' - \theta).$$

Since $\cos(-a) = \cos a$, the order of taking the factors is indifferent.

(*b*) **Vector product** of two vectors is defined as the product of their magnitudes and the sine of their included angle. The vector product of $\boldsymbol{F}$ and $\boldsymbol{S}$ will be written as $v\boldsymbol{FS}$ and is numerically equal to

$$FS \sin(\theta' - \theta)^* = rr' \sin(\theta' - \theta),$$

where F, S are the magnitudes of $\boldsymbol{F}$, $\boldsymbol{S}$, respectively. As the name indicates the vector product is a vector. The direction is that of the travel of a right-hand screw, perpendicular to the plane of the two vectors, when turned in such a way as to rotate the **first factor** toward the **second.** Then $\theta' - \theta$ is regarded positive. If the direction of rotation is reversed the angle $\theta' - \theta$ becomes negative and its sine becomes negative and consequently the vector product changes sign and its vector is reversed. The order of factors in the vector product is, therefore, not indifferent, but must be carefully noted.

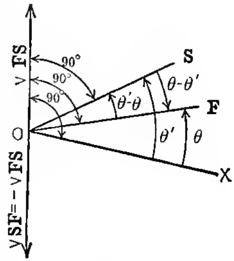


FIG. 66.

1. Find the scalar product of (3, 40°) and (6, 30°). Find the vector product, factors taken in order given.

Note: $30° - 40° = -10°$.

2. Find the scalar and vector products of (25, 35°) and (60, 124°) in order given.

3. If **work** is defined as the product of force multiplied by the displacement component in the direction of the force, show that work is the scalar product of force and displacement.

* It is noted that $S \sin(\theta' - \theta)$ is the perpendicular distance from the origin to a line through the terminal of $\boldsymbol{S}$ parallel to $\boldsymbol{F}$. This distance, in case $\boldsymbol{F}$ is a force, is called the moment arm of $\boldsymbol{F}$ about the origin as an axis. See Ex. 5 below.

4. What is the work done by a force of 2000 lbs. acting N. 30° E. in moving a car 50′ on a track due north? (No friction.)

5. The **moment** of a force about a point or axis is defined to be the product of the force by the perpendicular distance of the line of the force from the point or axis. Show that the moment of a force about any axis is the vector product of the vector distance from the line of the force to the axis and the vector force. Note the order of factors.

6. Find the moment of a force of 500 lbs. acting N. 40° E. about a point such that the vector from the point to the initial point of the force vector is 12 units in a direction N. 80° E.

7. The coördinates of the initial point of a force vector are (3, 6), the terminal point (4, 10). Find the moment of the force about the origin. The magnitude of the force is the length of the line joining the above two points. The moment arm is the distance of the line from the origin.

8. Find the scalar product of (16, 30°) by (24, 60°).

9. Find the vector product of the vectors in Ex. 8.

Note. — We shall, from now on, describe a vector having its initial point at the origin by giving the coördinates of its terminal point only.

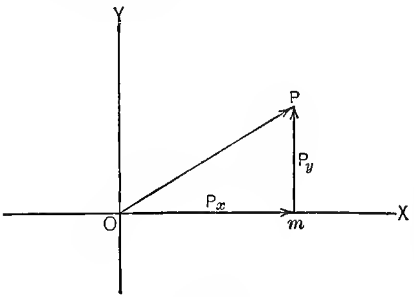


Fig. 68.

82. A vector may be regarded as the vector sum of its **components** on the axes. Thus the vector OP is the vector sum of Om and mP. We shall adopt the notation of writing a subscript to indicate the component. Thus the vector OP will be denoted by $\boldsymbol{P}$ and its component on OX by $\boldsymbol{P}_x$ and its component on OY by $\boldsymbol{P}_y$. The vector equation

$$\boldsymbol{P} = \boldsymbol{P}_x + \boldsymbol{P}_y$$

holds for all vectors.

The scalar and vector products of two vectors can now be expressed in rectangular coördinates. If $\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y$, $\boldsymbol{S} = \boldsymbol{S}_x + \boldsymbol{S}_y$ we may write:

(1) $$\boldsymbol{FS} = \boldsymbol{F}_x\boldsymbol{S}_x + \boldsymbol{F}_y\boldsymbol{S}_y.$$

Now if $\boldsymbol{F}$ is a force and $\boldsymbol{S}$ a displacement then $\boldsymbol{F}_x$ is a force and $\boldsymbol{S}_x$ a displacement in the same direction and $\boldsymbol{F}_x\boldsymbol{S}_x$ is the work done by $\boldsymbol{F}_x$ in the displacement $\boldsymbol{S}_x$. Similarly $\boldsymbol{F}_y\boldsymbol{S}_y$ is the work of $\boldsymbol{F}_y$ in the displacement $\boldsymbol{S}_y$. The right side of (1) then is the measure of work done by the components of $\boldsymbol{F}$. But work is the product of force and the component of displacement in the direction of the force, that is, the product of force and displacement and the cosine of their included angle. Therefore, the scalar product on the left of (1) represents the same thing that the right side represents and the two members of (1) are but different ways of expressing the scalar product of $\boldsymbol{F}$ and $\boldsymbol{S}$.

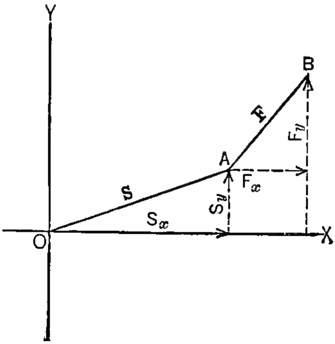


FIG. 69.

(2) $$v\boldsymbol{SF} = \boldsymbol{S}_x\boldsymbol{F}_y - \boldsymbol{S}_y\boldsymbol{F}_x, \text{ a vector.}$$

The term $\boldsymbol{S}_x\boldsymbol{F}_y$ is the moment of the force $\boldsymbol{F}_y$ acting at A and tending to swing A counterclockwise about O. The term $\boldsymbol{S}_y\boldsymbol{F}_x$ is the moment of $\boldsymbol{F}_x$ acting at A tending to swing A about O clockwise and is negative. Strictly speaking the right side of (2) is a scalar quantity, but owing to the known conventions as to the direction of rotation it contains the means of determining the direction of the tendency to rotation and we are justified in calling it a vector. The left side of (2) is known to be the moment of $\boldsymbol{F}$ about 0 when applied at A, **81**. Hence the two sides of (2) represent the same thing. We infer the generality of (1), (2), for all vectors. These products play an important rôle in physics and mechanics.

1. Find the axial components of (35, 26°); (125, −65°); (−275, −120°).

2. Find the sum of the components along the x-axis and the sum of the components along the y-axis in (1). Consider these sums as components of a new vector and calculate its modulus and amplitude (**75**).

3. Find the modulus and amplitude of the vector whose components are $\boldsymbol{P}_x = -496$, $\boldsymbol{P}_y = 275$.

4. Find the moment of $\boldsymbol{P}_x = 50$, $\boldsymbol{P}_y = 75$ applied at the point (10, 8).

5. If $\boldsymbol{F} = 30 + i\,50$ and $\boldsymbol{S} = 8 + 7\,i$, find the scalar and vector products of $\boldsymbol{F}$ and $\boldsymbol{S}$.

6. Reduce (25, 120°) and (80, 40°) to rectangular coördinates and find the scalar product and the vector product.

7. In (6) find the scalar and vector products without reducing to rectangular form.

8. Find the total moment about the origin of the vectors $(6 + 10\,i)$, $(-4 + 3\,i)$, applied at the point $(-3, 7)$.

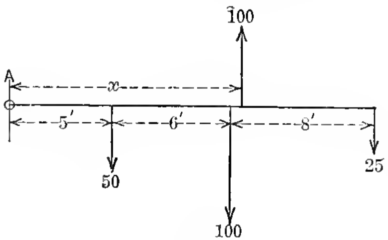


Fig. 70.

9. A lever has weights as shown in the diagram. Find the distance from A to the point of application of a 100 lb. force that will just balance the other forces.

Note. — The sum of the moments of the downward forces about A must equal the moment of 100 lbs. about A where x is the unknown moment arm. That is,

$$100\,x = 5 \cdot 50 + 11 \cdot 100 + 19 \cdot 25,$$

to find x.

83. The notation of vectors will now be applied to some problems in the **equilibrium** of **particles** and of **rigid bodies** acted upon by external forces.

A **particle** is a geometric point regarded as having inertia or mass.

A **rigid body** is one of finite fixed magnitude and unvarying form.

(*a*) **Equilibrium of a particle.**—In order that a particle shall not have its state of rest or of uniform motion in a straight line changed, all forces acting on the particle must be balanced. That is, there must be no component of resultant force along any line or axes of reference. Since the forces may be represented by vectors, this means the vector sum of all forces acting on the particle must be zero. This in turn means the polygon formed by the vectors in succession must be a closed polygon. This polygon is called the vector polygon of forces.

If the vector sum of the forces acting on a body is not zero there will be a resultant force equal to the closing side of the polygon which is the vector sum of the vectors of the forces. The calculation of the resultant force or vector sum of a set of forces is an important problem.

(*b*) A **rigid body** is in **equilibrium** when it is at rest; moving uniformly in a straight line, without rotation; rotating uniformly about a fixed axis or moving uniformly in a straight line and rotating about an axis whose direction is fixed.

For equilibrium of a rigid body (*a*) the conditions of equilibrium of a particle must be satisfied; (*b*) the sum of all moments acting on the body must be zero, when taken about any axis.

The solution of a problem relating to a rigid body is generally in two parts, viz.: First, consideration of the forces as acting on a particle of the same mass as the body; second, determination of the moments and the resultant moment acting on the body.

Some examples will illustrate how problems in equilibrium of particles and rigid bodies may be solved in ordinary cases.

I. Let a particle m be acted on by the forces $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, all in the same plane as shown in Fig. 71. To find the resultant force. $P = 75$, $Q = 100$, $R = 125$.

The vector equation for the resultant is:

$$\text{Resultant} = \boldsymbol{S} = \boldsymbol{R} + \boldsymbol{Q} + \boldsymbol{P}.$$

To calculate the magnitude and direction of the resultant, resolve along two perpendicular axes, mX and mY. Along mX,

$$S_x = S\cos\theta = R\cos 120° + Q\cos 10° + P\cos(-20°)$$

or

$$= 125\cdot(-0.5) + 100\cdot(0.985) + 75\cdot(0.939)$$

$$= -62.5 + 98.5 + 70.4.$$

$$\therefore\quad S_x = 106.4.$$

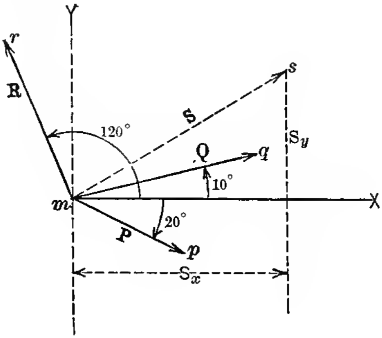


FIG. 71.

Along mY,

$$S_y = S\sin\theta = R\sin 120° + Q\sin 10° + P\sin(-20°)$$

$$= 125\cdot(0.866) + 100\cdot(0.173) + 75\cdot(-0.342)$$

$$= 108 + 17.3 - 25.6.$$

$$\therefore\quad S_y = 99.7.$$

Now

$$S = \sqrt{S_x^2 + S_y^2} = \sqrt{106.4^2 + 99.7^2} = 141.3$$

and

$$\tan\theta = \frac{S_y}{S_x} = \frac{99.7}{106.4} = 0.938.$$

$$\therefore\quad \theta = 43°\,10' \text{ (see figure, } ms = S).$$

In a similar manner an unknown force in any system acting on a particle can be determined if the equations of equilibrium can be written.

If the vectors are drawn carefully to scale the value of S can be found from the vector polygon. The method of construction can be seen from Fig. 65, § 80. Start at m and lay off the vectors in order. The closing side S is the resultant.

This method of solution is called the graphic method.

A third method, called the geometric method, is as follows: Draw the diagram and solve the problem by methods of Chapter VIII. See Figs. 72, 73.

First find angle α from the known directions of R and P. Then AC can be found. Having AC, determine angle BAC and angle CAO. Now OC and angle β can be found.

Let the student carry out the work.

It is strongly recommended that every problem be solved by two of the three methods. The graphic method may very well be used as one in each case.

II. Consider the case of a trap door as shown in the figure. It is desired to find the pull (tension) in the rope and the hinge reaction in the position given in Fig. 73.

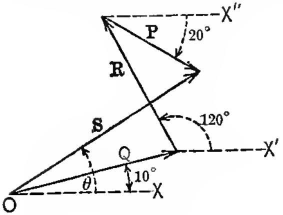


FIG. 72.

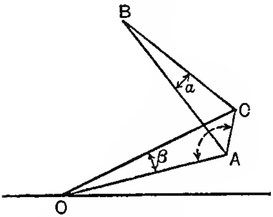


FIG. 73.

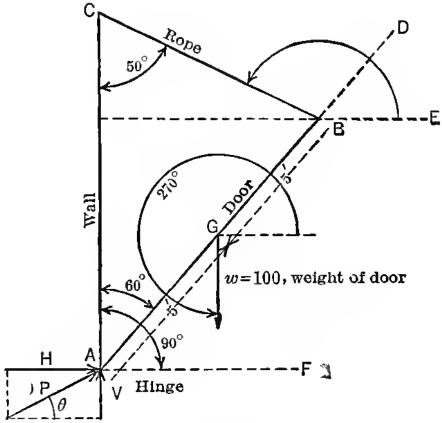


FIG. 74.

It is seen at first that the weight of the door regarded as applied at its center of gravity, G, tends to produce clockwise rotation about A, the hinge. Further the rope applied at B tends to produce counterclockwise rotation about A.

The measure of tendency to produce rotation by a force is the moment of the force. For equilibrium of the trap door the sum of the moments about A must equal zero. That is, the

two moments must be equal but in opposite directions, that is, of opposite sign. These two moments must be calculated.

The moment due to the pull, T, in the rope, is equal in magnitude to

$$\begin{aligned} M &= T \cdot AB \cdot \sin DBC \text{ (a magnitude of the vector product)} \\ &= T \cdot AB \sin CBA \\ &= T \cdot 10 \cdot \sin 70^\circ = 9.39\,T \text{ (counter clockwise).} \end{aligned}$$

The moment of the weight of the door is

$$\begin{aligned} M_2 &= w \cdot AG \sin CAG \text{ (magnitude of vector product)} \\ &= 100 \cdot 5 \cdot \sin 60^\circ \\ &= 433 \text{ (clockwise).} \end{aligned}$$

$$\therefore \quad 9.39\,T = 433$$

and $\qquad T = 46.1$ lbs., the tension in the rope.

To find the hinge reaction at A, we find its horizontal and vertical components. By taking horizontal components of all forces, calling the horizontal component H,

$$T \cos EBC + H \cdot \cos 0^\circ = 0,$$

$$46.1 \cos 140^\circ + H = 0,$$

whence $\qquad H = 35.4.$

By taking vertical components,

$$T \sin EBC + V \sin 90^\circ + W \sin 270 = 0,$$

$$46.1 \sin 140^\circ + V - 100 = 0,$$

whence $\qquad V = 70.4.$

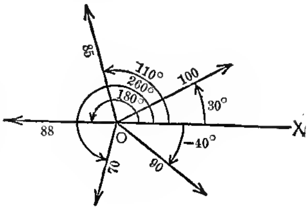
Now if P is the hinge reaction

$$P = \sqrt{(70.4)^2 + (35.4)^2} = 79.1, \text{ approximately,}$$

$$\tan\theta = \frac{70.4}{35.4} = 2, \text{ nearly,}$$

$$\theta = 63^\circ\,25', \text{ approximately.}$$

Thus the problem is completely solved.

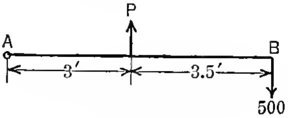


Ex. 1.

1. The forces shown in the diagram act on a particle. Calculate the magnitude and direction of the resultant.

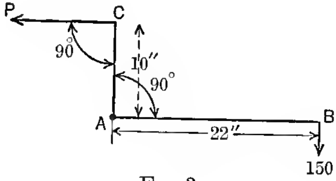
Note. — Take horizontal and vertical components as in 1.

2. Find the force P required to hold the lever AB in position as shown in the figure.



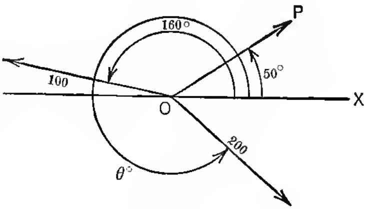
Ex. 2.

3. Find P so that the load at B will be sustained, CAB being a bent lever with fulcrum at A.



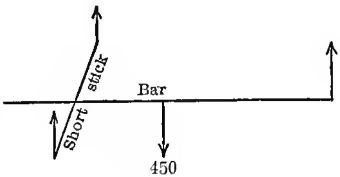
Ex. 3.

4. Find a force P making an angle 50° with OX so that the system will be held in equilibrium. Determine also the unknown angle, θ.



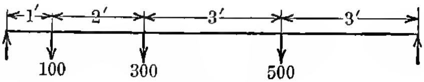
Ex. 4.

5. Three men carry a heavy uniform bar weighing 450 lbs., one man at one end, the other two with a short stick some distance from the other end. If the length of the bar is l, find how far from the end the two must lift so all lift the same amount.



Ex. 5.

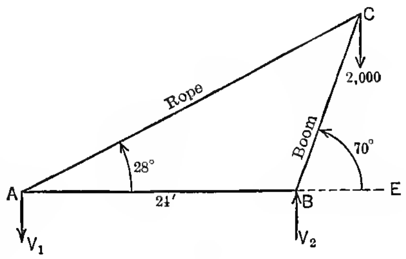
6. Find the supporting forces at the ends of the horizontal bar loaded as shown in the figure.



Ex. 6.

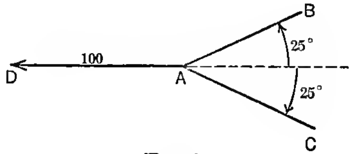
7. Find the stress in the rope of the crane shown in the figure. Find the forces V_1, V_2, considering ABC as a single rigid body acted upon by V_1, V_2 and 2000 lbs.

Note. — Start by taking resolutions at C, then moments at A.



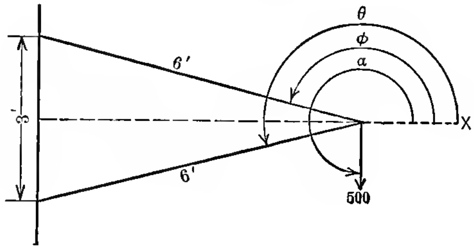
Ex. 7.

8. Three boys pull on three ropes tied to a ring as shown in the figure. Determine the pulls in ropes AB, AC. (Resolutions.)



Ex. 8.

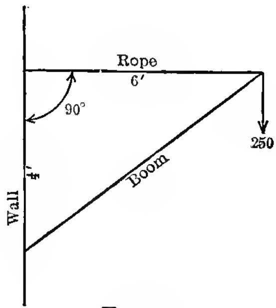
9. Determine the forces in the frame as shown in the figure.



Ex. 9.

10. What pull on the rope is necessary to hold the boom in position with the load as indicated in the figure.

11. Find the resultant of two forces of 500 lbs. each, one acting due north and the other N. 60° E.



Ex. 10.

CHAPTER X

EQUATIONS

84. Let $f(x)$ be defined by the expression:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where n is an integer and the a's are known constants. A function of this kind is called an **integral function** of the variable x, of degree n.

Theorem I. — If r is a root of the equation:

$$f(x) = 0,$$

then $f(x)$ is exactly divisible by $x - r$.

For divide $f(x)$ by $(x - r)$ by the ordinary method, continuing the process until a remainder not containing x is obtained, the result can be represented as

$$(1) \qquad f(x) = (x - r)\, q(x) + R,$$

where $q(x)$ denotes the quotient and R the remainder. If r is a root of $f(x) = 0$, the left side vanishes for $x = r$. The first term on the right also vanishes for $x = r$. Then (1) becomes

$$(2) \qquad 0 = 0 + R$$

whence $R = 0$. Therefore, the division is exact and

$$(3) \qquad f(x) = (x - r)\, q(x).$$

85. Assumption. — Every integral equation * has at least one root.

Theorem II. — Every integral equation of degree n has n roots. Write the equation as

$$(4) \qquad f(x) = 0.$$

* That is an integral function equated to zero.

This equation has a root, say r_1. By Theorem I:

(5) $$f(x) = (x - r_1) f_1(x) = 0,$$

where $f_1(x)$ denotes the quotient. Now as above

(6) $$f_1(x) = 0.$$

If $f_1(x)$ is not a constant it is an integral function of x of degree $n - 1$, and Eq. 6 has a root, say r_2. Hence

(7) $$f_1(x) = (x - r_2) f_2(x).$$

Continue this process until a quotient $f_n(x)$ not containing x is obtained. No more divisions can be carried out and no more roots exist. The result is that $f(x)$ has been broken up into factors as shown in the equation,

(8) $$f(x) = a_0 (x - r_1)(x - r_2) \ldots (x - r_n) = 0.$$

The values $r_1, r_2, \ldots, r_n$ are the only values of x which satisfy this equation and consequently the only roots of equation (4).

Incidentally equation (8) shows how to form an equation that shall have given roots.

86. Theorem III. — If the equation $f(x) = 0$ has more than n roots it is an identity and has infinitely many roots and every coefficient is zero.

Consider,

(9) $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

and suppose it has more than n roots. If every a is not zero the terms whose coefficients do not vanish will form an equation of degree not higher than n which can have not more than n roots. But this contradicts the hypothesis. Hence the theorem is true.

This theorem has important uses in mathematics, some of which will appear in the sequel.

Determine which are identities and which are not.

1. $(x - a)^2 = x^2 - 2\,ax + a^2$, expand left side, transpose and collect terms.

2. $x^2 - 4 = 0$. For how many values of x is this equation satisfied?

3. $\dfrac{x^2 - 9}{x + 3} = x - 3.$ Clear of fractions.

4. $x^3 - 3x^2 - x = 0.$

5. $\dfrac{1}{1 - x} = 1 + x + x^2 + \dfrac{x^3}{1 - x}.$

6. Is 2 a root of $x^2 + 4x + 4 = 0$? Is -2 a root?

7. Is 1 a root of $x^3 - 3x^2 + 3x + 1 = 0$? Is 2 a root?

87. Theorem IV. — If $f(x)$ be divided by $(x - r)$, the remainder will be $f(r)$.

For write $\qquad f(x) = (x - r)\,q(x) + R.$
Now put $x = r \qquad f(r) = (r - r)\,q(r) + R.$
Since $r - r = 0, \qquad R = f(r).$

This theorem furnishes a convenient method of calculating $f(r)$. To shorten the work it will be desirable to learn an abbreviated method of performing the division by $x - r$. This will be given later.

88. Zero and infinite roots of integral equations. — Consider

(1) $\qquad f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$

If $a_n = a_{n-1} = \cdots = a_{n-k+1} = 0$, k roots of (1) are zero.

Under this hypothesis (1) may be written,

(2) $\qquad a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$
$$= x^k(a_0x^{n-k} + a_1x^{n-k-1} + \cdots + a_{n-k}) = 0.$$

It is seen now that k roots are zero since $x^k = (x - 0)^k$ is a factor of the expression constituting the left member.

Now substitute $x = \dfrac{1}{y}$ in (1) and obtain, after multiplying by y^n,

(3) $\qquad a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0.$

If in (3) $a_0 = a_1 = a_2 = \cdots = a_{k-1} = 0$, k roots are zero, as shown above. But by virtue of the relation, $x = \dfrac{1}{y}$, $x = \infty$ when $y = 0$. Therefore, there is an infinite root of (1) for each

zero root of (3). Hence when the first k coefficients of (1) approach zero, k of its roots become infinitely large.

Zero roots often occur in practical work, and infinite roots have important meanings in certain types of problems.

1. What is the value of x in $xy = k$, when $y = 0$? If this equation is regarded as the equation of a curve what is the interpretation for $y = 0$?

2. What is the value of $y = x^2 - 2x + 1$, when $x = 1$? Interpret this result as in Ex. 1.

89. Synthetic division is a shortening of the process of long division when the divisor is a binomial of the first degree in the variable. Consider the example:

$$\begin{array}{rl}
x^3 - 4x^2 - 7x + 10 & \underline{|x - 5} \\
\underline{x^3 - 5x^2 \qquad\qquad\quad} & \underline{|x^2 + x - 2} \\
x^2 - 7x \qquad\quad & \\
\underline{x^2 - 5x} \qquad\quad & \\
-2x + 10 & \\
\underline{-2x + 10} &
\end{array}$$

If the cancelled terms are omitted the work appears as

$$\begin{array}{rl}
x^3 - 4x^2 - 7x + 10 & \underline{|x - 5} \\
\underline{- 5x^2 \qquad\qquad\quad} & \underline{|x^2 + x - 2} \\
x^2 - 7x \qquad\quad & \\
\underline{- 5x} \qquad\quad & \\
-2x + 10 & \\
\underline{+ 10} &
\end{array}$$

By pushing up the remainders into a line under the dividend the work appears as

$$\begin{array}{rl}
x^3 - 4x^2 - 7x + 10 & \underline{|x - 5} \\
\underline{- 5x^2 - 5x + 10} & \underline{|x^2 + x - 2} \\
x^2 - 2x \qquad 0 &
\end{array}$$

The x's may be omitted, using coefficients only. Thus

$$\begin{array}{rl}
1 - 4 - 7 + 10 & \underline{|-5} \\
\underline{- 5 - 5 + 10} & \underline{|1 + 1 - 2} \\
1 - 2 \qquad 0 &
\end{array}$$

It is noticed that the last form gives all the information given in the first form of division. If the coefficient of the first term of the dividend is brought down, then the partial remainders in order are the coefficients of the quotient and the final remainder is zero.

In applying this method it is desirable to use addition instead of subtraction during the process. This is done by changing the sign of the second term of the divisor, that is change -5 to 5. Thus,

$$\begin{array}{rrrr|l} 1 & -4 & -7 & +10 & \underline{5} \\ & +5 & +5 & -10 & \\ \hline 1 & +1 & -2 & 0 & \end{array}$$

90. Theorem V. — If $f(x)$ is exactly divisible by $x - r$, r is a root of the equation $f(x) = 0$.

This theorem is proved by means of the equation used to prove Theorem I. For if $R = 0$,

$$f(x) = (x - r)\,q(x).$$

Substituting r for x causes the right side to vanish, and hence the left side also. Therefore, r is a root of $f(x) = 0$, by definition of root.

It is now easy to use Theorem V and synthetic division to determine whether or not any given number is a root of a given integral equation. If a number should not be a root it is useful to know that the final remainder R is the value of $f(r)$.

In applying synthetic division, if any term in x is missing its place must be filled with a zero.

Solve by synthetic division:

1. Is 2 a root of $x^3 - x^2 - 3x - 3 = 0$?
2. Is 1 a root of $x^3 - x^2 + 3x - 3 = 0$?
3. Is $x - 3$ a factor of $3x^4 - 11x^3 + 5x^2 + 3x = 0$.
4. Factor $6x^2 + 19x + 10$.
5. Factor $x^3 + 5x^2 + 2x + 10$.
6. Factor $x^4 - 2x^3 - 8x - 16$ (supply term in x^2 by 0).

7. Find any integral root of $x^3 - 25x^2 + 8x - 16 = 0$.
8. Find any integral root of $2x^3 - 3x^2 - 3x + 2 = 0$.
9. Find any integral root of $x^4 + 2x^3 - 5x^2 - 4x + 6$.

91. Solution of numerical equations in one unknown of any degree. — No general simple rule can be given for finding the roots of equations of degree higher than the second. It is possible to solve equations of the third and fourth degrees by formulas, but the application is in general not easy. For practical purposes some method of approximation is most useful. Such a method can be easily constructed from the preceding theorems.

For values of x not roots of $f(x) = 0$, it is obvious that the result of substituting such values in $f(x)$ would give positive or negative values of the function instead of zero.

For values of x near r, where r is a root of $f(x) = 0$ and h a small positive number, one of the following relations must, in general, hold:

$$(a) \quad f(r-h) < f(r) = 0 < f(r+h),$$

or

$$(b) \quad f(r-h) > f(r) = 0 > f(r+h),$$

or

$$(c) \quad f(r-h) < f(r) = 0 > f(r+h),$$

or

$$(d) \quad f(r-h) > f(r) = 0 < f(r+h),$$

or

$$(e) \quad f(r-h) = f(r) = 0 = f(r+h).$$

Conditions (a) and (b) are the ones which will now be considered, being the most commonly occurring. These can be used to discover the position of the real roots of an equation of any form.

Consider first an integral equation,

$$f(x) = x^3 - 4x^2 - 2x + 8 = 0.$$

Note. — What follows illustrates a method. In practice one would first determine whether integral factors of 8 are roots.

Try, by synthetic division,* in succession the values -3, -2, -1, 0, 1, 2, 3, etc.:

* Synthetic division is here only a convenient method of substituting values of x in the equation.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-3} \\ & -3 & +21 & -57 & \\ \hline & -7 & +19 & -49 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & -2 & -8 & \underline{-2} \\ & -2 & +12 & -20 & \\ \hline & -6 & +10 & -28 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1} \\ & -1 & +5 & -3 & \\ \hline & -5 & +3 & +5 & \end{array}$$

$x = -3$, $R = f(-3) = -49$
$x = -2$, $R = f(-2) = -28$
$x = -1$, $R = f(-1) = +5$
By condition (a) a root lies between -2 and -1.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{0} \\ & 0 & 0 & 0 & \\ \hline & -4 & -2 & +8 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{1} \\ & 1 & -3 & -5 & \\ \hline & -3 & -5 & +3 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{2} \\ & 2 & -4 & -12 & \\ \hline & -2 & -6 & -4 & \end{array}$$

$x = 0$, $R = f(0) = +8$
$x = 1$, $R = f(1) = +3$
$x = 2$, $R = f(2) = -4$
By condition (b), a root lies between 1 and 2.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{3} \\ & 3 & -3 & -15 & \\ \hline & -1 & -5 & -7 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{4} \\ & 4 & 0 & -8 & \\ \hline & 0 & -2 & 0 & \end{array}$$

$x = 3$, $R = f(3) = -7$
$x = 4$, $R = f(4) = 0$
4 is a root.

Bringing down the first coefficient the quotient is $x^2 - 2$. Now having found one root, 4, exactly the equation can be written

$$x^3 - 4x^2 - 2x + 8 = (x - 4)(x^2 - 2) = 0$$

and the remaining roots are easily found from the equation

$$x^2 - 2 = 0.$$

But to illustrate the method, when an exact root is not found the fact that 4 is a root will be ignored and we shall proceed to find the root that lies between -2 and -1.

Take a value of x midway between -2 and -1, that is -1.5,

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1.5} \\ & -1.5 & +8.25 & -9.375 & \\ \hline & -5.5 & +6.25 & -1.375 & \end{array}$$

$x=-1.5$, $R=f(-1.5)=-1.375$. By condition (a) a root lies between -1.5 and -1.

Now take the value midway between -1.5 and -1. Since this value involves three figures, it will be better to take a near value in two figures as -1.2 or -1.3. Again, since the value of R for $x = -1.5$ is smaller, numerically, than the value of R for $x = -1$, we will risk choosing -1.3.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1.3} \\ & -1.3 & +6.89 & -6.36 & \\ \hline & -5.3 & +4.89 & +1.64 & \end{array}$$

$x = -1.3$, $R = f(-1.3) = 1.64$. Root between -1.5 and -1.3.

Now take -1.4, midway between -1.5 and -1.3.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1.4} \\ & -1.4 & +7.56 & -7.784 & \\ \hline & -5.4 & +5.56 & +0.216 & \end{array}$$

$x=-1.4$, $R=f(-1.4)=+0.216$. Root between -1.4 and -1.5.

Since $f(-1.4)$ is much smaller (numerically) than $f(-1.5)$ we will try values quite near -1.4. Take -1.41.

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1.41} \\ & -1.41 & +7.628 & -7.935 & \\ \hline & -5.41 & +4.628 & +0.065 & \end{array}$$

Take -1.42,

$$\begin{array}{rrrr|l} 1 & -4 & -2 & +8 & \underline{-1.42} \\ & -1.42 & +7.696 & -8.088 & \\ \hline & -5.42 & +5.696 & -0.088 & \end{array}$$

$x = -1.41$, $R = f(-1.41) = 0.065$

$x = -1.42$, $R = f(-1.42) = -0.074$.

Root between -1.41 and -1.42.

The root is, therefore, 1.41 correct to three figures. The nearly equal values of R for -1.41 and -1.42 would suggest that the next figure is near 5 and a continuation of the work will show that the next figure of the root is 4 and the root is -1.414 correct to four figures. This value is sufficiently exact for all ordinary purposes.

The above process is rather laborious. Practice will enable one to obtain three or four figures of a root quite readily. None

of the current methods of finding irrational roots are much shorter, if any. Some are longer and require more theoretical knowledge.

1. Find the roots of $x^4 + x^3 - 7x^2 - 5x + 10 = 0$.
2. Find the roots of
$28x^4 + 239x^3 + 1020x^2 + 813x - 140 = 0$.
3. Find the root of Ex. 13, following **34**, by the above method.

Consider another type of equation,

$$x - \sin x - 1 = 0.$$

Synthetic division cannot be employed here as a method of substituting values of x, for the reason that this equation, as will be shown later, has an infinite number of terms when we expand it into an integral equation. Such an equation is called transcendental. In this case the angle, x, in the first term must be given in radians. The values of $\sin x$ are to be taken from a table of natural sines.

Try $x = 2$ radians $= 114° 35'$, nearly.

$2 - 0.9095 - 1 = +0.0905$, $x = 2$ rad., $R = f(2) = 0.0905$.

If $x = 1.9$ radians $= 108° 51'$

$1.9 - 0.9463 - 1 = -0.0463$, $\left.\right\}$ $x = 1.9$ rad., $R = f(1.9) = -0.0463$. Root between 1.9 and 2 radians.

If $x = 1.95$ radians

$1.95 - 0.9291 - 1 = 0.0209$, $x = 1.95$ rad., $R = f(1.95) = 0.0209$.

If $x = 1.93$ radians

$1.93 - 0.9362 - 1 = -0.0062$, $x = 1.93$ rad., $R = f(1.93) = -0.0062$.

This value is correct within an error of less than angle of $10'$. Further trials will give more accurate results.

Solve by repeated substitution:

1. $x = \tan x$.
2. $x = 3\sin x - 1$.
3. $3x + \log_{10} x = 5$.
4. $\sin x + \cos x = 1.4$.

Note. — In some cases it may be useful to construct the graph of the equation before attempting the solution. The intercept of the graph on the axis of abscissas will furnish a guide in selecting trial values for the root.

5. The distance from the earth of a body projected vertically upward is given by the formula

$$s = v_0 t - 16.1\, t^2,$$

where v_0 is the velocity of projection upward in feet per second and t is the time from starting, in seconds and s is the distance in feet. If a body is projected upward with a velocity of 80′/sec., in how many seconds from starting will it be 30′ from the earth? (Two values of t.)

6. A stone is dropped into a well. It is heard to strike the bottom after 5 sec. Having given $s = 16.1\, t^2$ and the velocity of sound $= 1150'$/sec., determine the depth of the well.

7. The volume of a cubical box is diminished 1200 cu. in. by putting in it 1/2 in. lining on all faces. Find the original dimensions of the box.

8. The compound amount of \$3000 at x per cent, for 5 yrs., is \$3500. Find the rate x. Given $A = P\left(1 + \frac{x}{100}\right)^t$.

9. Find the length of a chord of a circle that cuts off $\frac{1}{3}$ its area if the radius is 1.

10. Find the x-intercepts of the curve whose equation is

$$y = 2\,x^3 - x^2 - 6\,x + 3.$$

Note. — Write the function equal to zero and solve the equation for its roots.

11. How deep will a sphere of wood 1′ in diameter sink in water if the density of the wood is 0.7 that of water, given the volume of a spherical sector equals the area of its spherical surface times $\frac{1}{3}$ its radius and the volume of a cone equals $\frac{1}{3}$ its base times its altitude.

12. Find the real fifth root of 15.

Note. — $x^5 - 15 = 0$. Solve this equation in the regular way.

13. In the solution of a certain problem in mechanics the result depended on solving the equation

$$\cos^3\theta + 0.8\cos^2\theta - 0.02\cos\theta - 0.393 = 0.$$

Determine one root to three figures. ($\cos\theta$ as unknown.)

92. Particular case of quadratic equation. — The frequent occurrence of equations of the second degree (quadratic equations) makes it desirable to give them some special treatment. All the theorems regarding integral equations, given in this chapter so far, hold for quadratic equations.

By Theorems II, III a quadratic equation has two and only two roots. It can, therefore, be written in the form

$$k(x - r_1)(x - r_2) = 0, \tag{1}$$

where k is constant and r_1, r_2 are the roots. Equation (1) can be written

$$kx^2 - k(r_1 + r_2)x + kr_1r_2 = 0 \tag{2}$$

or

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0.$$

The equation (2) is of the form

$$ax^2 + bx + c = 0 \tag{3}$$

or

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Comparing the left-hand members of (2) and (3) it is easily seen that if they represent the same equation

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1r_2 = \frac{c}{a}.$$

The formula given in **10** may be obtained as follows:

Given

$$ax^2 + bx + c = 0.$$

Dividing by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Adding $\frac{b^2}{4a^2}$ to both members,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of the first and last members,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The two roots of the quadratic will be equal to each other if $b^2 - 4ac$ * $= 0$. In this case each root is $-b/2a$.

What will be the nature of the roots if $b^2 - 4ac < 0$?

What will be the nature of the roots if $b - 4ac > 0$?

By calculating the expression $b^2 - 4ac$ it is possible to know the nature of the roots of the quadratic equation without actually calculating them. Thus, if

$$x^2 + x + 1 = 0,$$

then,

$$a = 1, \quad b = 1, \quad c = 1, \quad b^2 - 4ac = -3.$$

Therefore the roots are complex and unequal.

Determine the nature of the roots of the following:

1. $4x^2 + 8x + 4 = 4.$
2. $3x^2 - 6x + 3 = 0.$
3. $x^2 - 7x + 16 = 0.$
4. $9x^2 - 12x - 6 = 0.$

5. If $c = 0$, one root of the quadratic is 0, **88.** If $b = 0$, the roots are equal numerically but of opposite signs. They are given by $\pm\sqrt{\frac{-c}{a}}$.

93. Equations of higher degree than the second may sometimes be written in quadratic form. The following is such an equation,

$$2x^6 - 5x^3 + 2 = 0.$$

* This expression is called the discriminant of the quadratic.

First consider x^3 as the unknown, then the equation is a quadratic. Solving by the formula of the preceding section

$$x^3 = \frac{5 \pm \sqrt{25-16}}{4} = 2, \text{ and } \frac{1}{2}.$$

The original equation can now be written in the form

$$(x^3 - 2)(x^3 - \tfrac{1}{2}) = 0.$$

It is necessary now to solve the two equations

$$x^3 - 2 = 0, \quad x^3 - \tfrac{1}{2} = 0.$$

Whence,

$$(x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}\,x + \sqrt[3]{4}) = 0;$$
$$(x - \sqrt[3]{\tfrac{1}{2}})(x^2 + \sqrt[3]{\tfrac{1}{2}}\,x + \sqrt[3]{\tfrac{1}{4}}) = 0.$$

From the second factor of the first equation, by the formula

$$(a) \qquad x = \frac{-\sqrt[3]{2} \pm \sqrt{\sqrt[3]{4} - 4\sqrt[3]{4}}}{2}.$$

The approximate values of the radicals should be substituted and the values of x expressed in usable form. From the first factor of the same equation

$$(b) \qquad x = \sqrt[3]{2}.$$

From the second factor of the second equation,

$$(c) \qquad x = \frac{-\sqrt[3]{\tfrac{1}{2}} \pm \sqrt{\sqrt[3]{\tfrac{1}{4}} - 4\sqrt[3]{\tfrac{1}{4}}}}{2}.$$

From the first factor of the same equation,

$$(d) \qquad x = \sqrt[3]{\tfrac{1}{2}}.$$

There are, altogether, six roots; these should be expressed as decimals to four figures. The solution of this problem should be carefully mastered as a typical case of higher equations in quadratic form. Use table I for evaluating radicals.

1. $x^4 - 6x^2 + 6 = 0.$
2. $x^{\frac{3}{2}} + 2x^{\frac{3}{4}} + 1 = 0.$
3. $x^2 - 3x + 2\sqrt{x^2 - 3x + 2} = 1.$
4. $\sqrt{x-1} - 5\sqrt[4]{x-1} = -6.$

CHAPTER XI

THE LINEAR FUNCTION AND THE STRAIGHT LINE

94. The **most general function** of the first degree in one variable is of the form:

$$f(x) = mx + b, \tag{1}$$

where m and b are constants. The most general equation of the first degree in two variables is of the form:

$$Ax + By + C = 0, \tag{2}$$

where A, B and C are constants. Solving (2) for y,

$$y = -\frac{A}{B}x - \frac{C}{B} \tag{3}$$

which is of the form

$$y = mx + b. \tag{4}$$

Equation (4) is of the same form as (1) except that y is written for $f(x)$. Equation (3) shows that (2) can be written in the form of (4). Since (3) is obtained from (2) by transposing and dividing by a constant, it is virtually the same equation. Since (4) is of the same form as (3), any conclusions drawn from (4) will be valid for (3) and consequently for (2).

95. Theorem. — The graph of any equation of the first degree in two variables is a straight line.

Using equation (4), let (x_1, y_1) be any fixed point on the locus or graph. Let (x_2, y_2) be any other point on the graph, chosen arbitrarily. Then by **34**,

$$y_1 = mx_1 + b,$$
$$y_2 = mx_2 + b.$$

Subtracting and solving for m,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since m is constant by hypothesis, Eq. (4), **94**, and since x_2, y_2 is any point different from (x_1, y_1) it is evident the slopes of the segments joining any two points of the locus are equal. This can be true only if the locus is a straight line.

If θ is the angle between the x-axis and the line, the slope of the line is

$$\tan \theta = m = \frac{y_2 - y_1}{x_2 - x_1},$$

where (x_1, y_1), (x_2, y_2) are any two points on the line. The angle, θ, is called the inclination of the line with the axis.

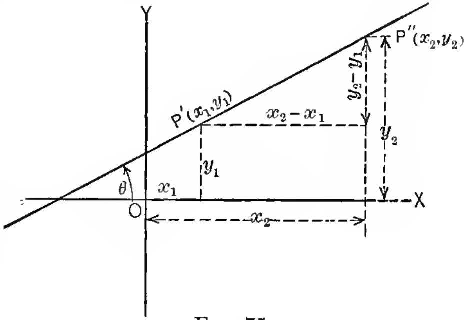


Fig. 75.

Note that the slope is the coefficient of x in the form (4), **94.**

1. Determine the slope, intercepts and draw the lines represented by the following equations:

a. $2x + 3y - 1 = 0.$

b. $x - y = 14.$

c. $x + y = -14.$

d. $\dfrac{x}{100} + 100y = \dfrac{1}{100}.$

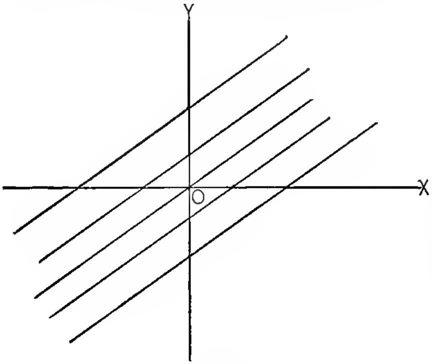


Fig. 76.

96. If in Equation (4), m remains fixed while b varies, there results an infinite number of equations representing an infinite number of parallel lines covering the whole plane. A set of lines having a common property is called a **family** of lines.

If b remains fixed while m varies there results a family of lines passing through the same point $(0, b)$, on the y-axis.

When a coefficient, which is ordinarily constant in an equation, is made to vary in the manner above indicated, it is called a **parameter.** The family of lines obtained by means of one such parameter is called a one-parameter family.

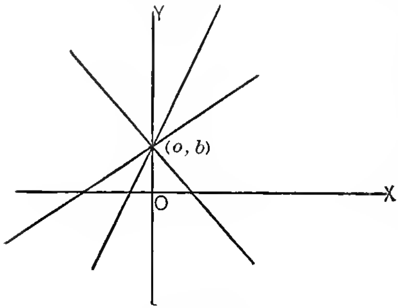


FIG. 77.

Draw several lines from each of the equations below by assigning different values to the parameter.

$$y = 2x + b.$$

Give b different values and draw a line for each value of b.

$$y = mx + 4.$$

Give m different values and draw a line for each.

97. Converse theorem. — Every straight line is represented by an equation of the first degree in two variables.

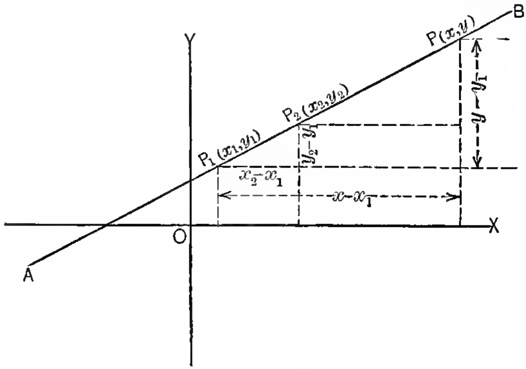


FIG. 78.

Let (x_1, y_1) and (x_2, y_2) be any two given points on the line. Let (x, y) be any third point on the line. The slope of $P_1 P$ is

$$\frac{y - y_1}{x - x_1}.$$

The slope of P_1P_2 is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Since the three points are on a straight line the two slopes just written are equal. Hence

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

(5) or
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This equation is of the first degree in the variables x and y Hence the theorem is true.

1. Find the equation of the straight line through $(-2, 1)$ and $(5, -4)$. Using (5) above

$$y - 1 = \frac{-4 - 1}{5 - (-2)}(x - (-2)),$$

or
$$y - 1 = -\tfrac{5}{7}(x + 2).$$
$$7y + 5x + 3 = 0.$$

2. Find the equation of the line through (1, 3) and (2, 5).

3. The slope of a line is 2. It passes through the point (3, 7). Find its equation.

Note. — Use equation (5), noting that

$$\frac{y_2 - y_1}{x_2 - x_1} = m = \text{slope}.$$

4. The slope of a line is m. It passes through the point (x_1, y_1). Find its equation.

5. Is the point (6, 4) on the line $y = 4x - 2$?

6. Are the points (3, 1), (4, 3), (6, 8) on the same straight line?

7. Find the equation of the line through (5, 7) and having a slope of $-\frac{1}{3}$.

8. Find the equation of the line through (1, 2), $(-1, 6)$.

9. Find the equation of the line through $(3, -5)$ and parallel to the line $3x - 4y + 2 = 0$.

10. What is the equation of the line parallel to the x-axis and distant 4 units from it? (Two solutions.)

98. Let (x, y) be the coördinates of any point P on the line AB and $p = Oe$ the distance of the line from O. Let the angle TOe be α. Now the projections of x and y along Oe are such that

(6) $$x \cos \alpha + y \sin \alpha = p.$$

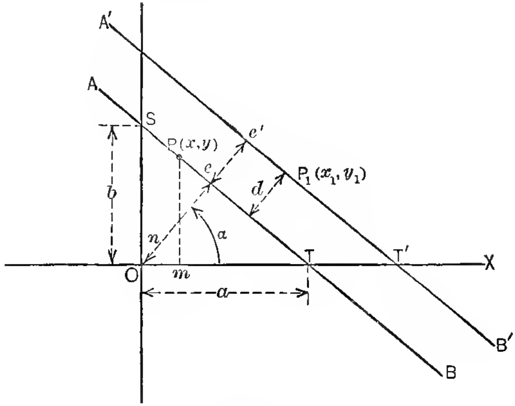


FIG. 79.

This relation holds for all points on AB. Therefore (6) is the equation of the line AB. Equation (6) is called the normal form of the equation of a straight line. This form is useful in problems relating to the distance of a point from a line.

Let $A'B'$ be a line parallel to AB, and let (x_1, y_1) be any point on $A'B'$. Then if $p_1 = Oe'$,

(a) $$x_1 \cos \alpha + y_1 \sin \alpha = p_1 = p + d,$$

where d is the distance of (x_1, y_1) from AB, or the distance between the lines. From (a) by transposing,

(b) $$x_1 \cos \alpha + y_1 \sin \alpha - p = d.$$

The left side of (b) is exactly what (6) becomes if p is transposed to the left side and (x_1, y_1) substituted for (x, y). Hence the distance of any point (x', y') from a line AB may be found by reducing the equation of AB to normal form and transposing all terms to the left side, then substituting the value (x', y') for (x, y). The result is the distance of the point (x', y') from

AB. This distance is to be considered positive if the point (x_1, y_1) and the origin are on opposite sides of the line AB, and negative if the point (x_1, y_1) and the origin are on the same side of the line AB.

From the triangle SOT, $p = b \sin \alpha = a \cos \alpha$, and $a/b = \tan \alpha$. From **47** it is seen that

$$\sin \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \ \cos \alpha = \frac{b}{\sqrt{a^2 + b^2}}.$$

By **94**, Eq. 3, since a and b are the intercepts, if the equation of a line is

$$Ax + By + C = 0,$$

then

$$a = -C/A, \quad b = -C/B, \quad p = \frac{ab}{\sqrt{a^2 + b^2}}.$$

It follows that

$$\cos \alpha = \frac{-C/B}{\pm\sqrt{C^2/A^2 + C^2/B^2}} = \frac{-A}{\pm\sqrt{A^2 + B^2}},$$

$$\sin \alpha = \frac{-C/A}{\pm\sqrt{C^2/A^2 + C^2/B^2}} = \frac{-B}{\pm\sqrt{A^2 + B^2}},$$

$$p = \frac{-C}{\pm\sqrt{A^2 + B^2}}.$$

Thus if $Ax + By + C = 0$ is the general form of the equation of a straight line, then

$$\text{(7)} \qquad \frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y = \frac{-C}{\pm\sqrt{A^2 + B^2}}$$

is the normal form of the equation of the same line. It is customary to choose the sign of the radical in (7) so that the right member of the equation shall be positive.

99. The **different forms** of the equation of the straight line in common use may be summarized as follows:

1. $Ax + By + C = 0$, **general form.**
2. $y = mx + b$, **slope-intercept** form.
3. $y - y_1 = m(x - x_1)$, **one-point slope** form (see Ex. 4, **97**).

4. $y - y_1 = [(y_2 - y_1)/(x_2 - x_1)]\,(x - x_1)$, **two-point** form.

5. $x/a + y/b = 1$, **intercept** form.

To obtain (5), divide (1) by $-C$ and note that the values of a, b above are the intercepts of the line.

6. $$\left.\begin{array}{l} x\cos\alpha + y\sin\alpha = p \quad \text{or} \\ \dfrac{A}{\pm\sqrt{A^2+B^2}}x + \dfrac{B}{\pm\sqrt{A^2+B^2}}y = \dfrac{-C}{\pm\sqrt{A^2+B^2}} \end{array}\right\}\textbf{Normal}\text{ form.}$$

These forms are to be memorized. Their use is somewhat suggested by their names. In a problem, careful attention to what is given regarding a line will often suggest what form of the equation is to be used.

100. To find the **distance between two points,** having given their coördinates.

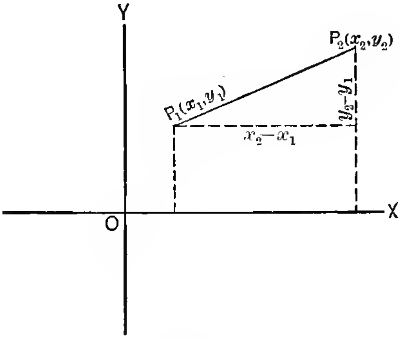


FIG. 80.

From the figure it is seen that **13,**

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This formula is true for points in all positions. Care must be used regarding the signs of the coördinates.

101. The **coördinates** of a point that **divides** the segment joining two points in a given **ratio** can be found as follows: Let AB be the segment and P the point of division and r the

ratio of the two parts into which AB is to be divided. From the similar triangles in the figure

$$\frac{AP}{PB} = \frac{AM}{PS} = r,$$

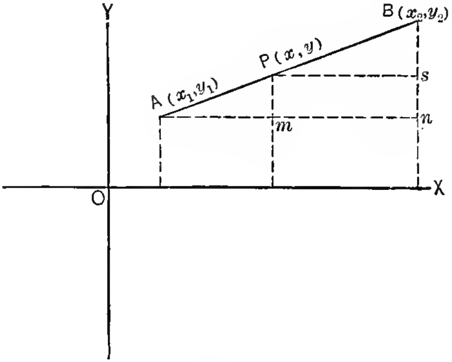


FIG. 81.

or since $AM = x - x_1$, $PS = x_2 - x$ the equation becomes

$$\frac{x - x_1}{x_2 - x} = r,$$

and

$$x = \frac{rx_2 + x_1}{r + 1}.$$

In a similar way,

$$y = \frac{ry_2 + y_1}{r + 1}.$$

If P is not between A and B the ratio r is negative. The line AB is then divided externally.

1. Find the coördinates of the points of trisection of the segment joining $A(2, 1)$ to $B(8, -4)$.

Note. — Since there are two points of trisection the solution is to be done in two parts. First let $r = \frac{1}{2}$, then

$$x = \frac{\frac{1}{2} \cdot 8 + 2}{1 + \frac{1}{2}} = 4$$

and similarly for y. For the second point call $r = 2$ and use the same formula. Student complete the solution.

2. Find the middle point of the segment in Ex. 1.

Note. — Call $r = 1$ and proceed as above.

3. Find the coördinates of the point nearest (2, 1) that divides the line in Ex. 1, externally in the ratio of $\frac{2}{5}$.

Note. — Call $r = -\frac{2}{5}$.

4. Find the coördinates of the point that divides the line in Ex. 1, externally in the ratio of 1 to 1. What is the geometric interpretation?

102. The **angle between two lines** can be determined from their slopes. It is evident that in the figure $\phi = \theta_2 - \theta_1$.

Hence
$$\tan \phi = \tan (\theta_2 - \theta_1)$$
$$= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1} = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

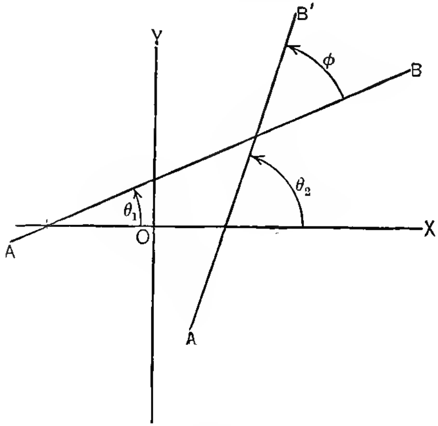


FIG. 82.

But $\tan \theta_2$ and $\tan \theta_1$ are the slopes of $A'B'$ and AB respectively. The slopes are to be found by any available method and substituted in the above equation. The result is the tangent of the angle ϕ. In applying this rule it is desirable to take for θ_2 the larger of the two angles θ_2 and θ_1 if it is possible to determine

which is the larger. Then ϕ is the angle through which AB must revolve in the positive direction (counter clockwise) to bring it into parallelism with $A'B'$.

1. Find the angle between the lines $2x - 3y + 4 = 0$ and $x - y = 1$.

Note. — Find the slopes from the equations and substitute the slopes in the above formula.

2. Find the angles of the triangles whose vertices are (1, 1); (6, 8); (7, −3).

Note. — From the coördinates of the points in pairs find the slopes of the lines joining them and proceed as in Ex. 1.

3. What relation must hold between $\tan\theta_2$ and $\tan\theta_1$ in order that $A'B'$ shall be parallel to AB?

4. Knowing that $\tan 90° = \infty$, show that AB will be perpendicular to $A'B'$ if $\tan\theta_2 \cdot \tan\theta_1 = -1$.

5. Show that the figure whose vertices are (0, 1); (2, 0); (5, 6); (3, 7) is a rectangle.

MISCELLANEOUS EXERCISES

1. What is the slope of each of the following lines?

(*a*) $10y + 3x = 6$; (*b*) $Ax + By + C = 0$; (*c*) $\frac{x}{6} - \frac{y}{7} = 1$.

2. Find the equations of the lines satisfying the conditions given below and keep the results for later use. Draw each line.

(*a*) Through the point (2, 1) with a slope of 2.
(*b*) Through the point (−3, 4) with a slope of $-\frac{3}{4}$.
(*c*) Through the points (1, 1) and (0, 2).
(*d*) Through the points (−4, 1) and (3, 8).

3. Find the equation of the line through (1, 5) and parallel to the line in (*c*) above.

4. Find the equation of the line through (2, 3) and at a distance $2\frac{1}{2}$ from the origin. Draw the line or lines.

5. Find the equation of the line through (−3, 1) and at a distance 3 from the origin. Draw the line or lines.

6. What is the normal equation of the line in (*b*) above?

7. What is the equation of the line whose intercepts are $a = 3$, $b = 1$?

8. Reduce all the equations obtained in 2 to intercept form.

9. What is the angle between the lines, $2x - 3y - 1 = 0$ and $x - 4y = 3$?

10. What is the angle between the lines $x/8 - y/2 = 1$ and $x \cos 30° + y \sin 30° = 4$?

11. What is the distance of (1, 2) from each line in Ex. 10?

12. What is the normal equation of the line whose intercepts are $a = -2$, $b = 8$?

13. The vertices of a triangle are (1, 2); (−3, 4); (4 7). Find the perimeter. Save all results.

14. What is the altitude of each ertex of the triangle in Ex. 13?

15. What is each angle of the triangle in Ex. 13?

16. What is the equation of the line through the origin and making a 45° angle with the x-axis?

17. What is the equation of the line through (3, 5) and parallel to the line $3x - 4y = 2$?

18. What is the equation of the line through 3, 5 and perpendicular to the line in Ex. 17?

19. Find the coördinates of the points which divide the segment joining (−8, 4) to (3, −5) into four equal parts.

20. What are the coördinates of the point midway between (3, 5) nd the line $y - 2x + 8 = 0$.

21. Show that the bisector of an angle of a triangle cuts the opposite side into segments having the same ratio as the sides adjacent the bisected angle.

22. Find the equation of the locus of all points equally distant from the points (2, 3) and (4, 1).

23. A square field has a tree near its center. Its distances from three corners are 7 rds., 8 rds. and 13 rds., respectively. Find the side of the field.

24. A regular pentagon has one vertex at the origin and one side in the x-axis. The length of a side is 24. Find the equations of the lines in which all the sides lie, respectively.

CHAPTER XII

EQUATIONS OF THE SECOND AND HIGHER DEGREES AND THEIR GRAPHS

103. An equation is said to be **explicit** for y if it is solved for y in terms of the other variables and constants. If x and y are the variables in the equation, it is explicit for y if it is of the form

$$y = f(x),$$

where $f(x)$ is any function of x. The forms

$$x^2 + y^2 = 25 \quad \text{and} \quad x^3 + 3\,xy + 4 = 0$$

and more generally

$$F(x, y) = 0,$$

are called **implicit** equations or functions in x and y. In such cases y is said to be an implicit function of x and x an implicit function of y, as is most convenient. By solving the equation for y or for x it becomes explicit.

104. Explicit quadratic function. — $y = ax^2 + bx + c$. The graph of an equation of this form was constructed in **34.** Another example will now be discussed. Consider

$$y = x^2 + 2\,x + 3.$$

$$\begin{array}{rrrrrrrrr} x = & -4, & -3, & -2, & -1, & -0, & 1, & 2, & 3. \\ y = & 5, & 0, & -3, & -4, & -3, & 0, & 5, & 12. \end{array}$$

The graph is shown in Fig. 83. Answer the **following questions:**

1. Does the curve pass through the origin? How is this determined from the equation?

2. Determine the intercepts on both axes.

3. Does any branch of the curve extend indefinitely from the origin? That is, does the curve have infinite branches?

4. Is the curve a closed curve?

5. Is the curve symmetrical about either axis or about any other line or about the origin?

6. What values, if any, of either variable must be excluded? That is, what values do not correspond to points on the curve?

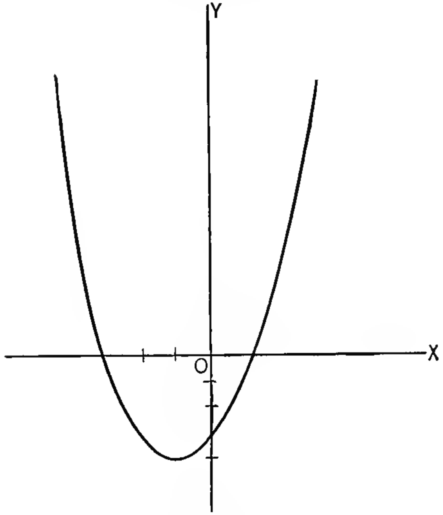


FIG. 83.

These questions will now be answered with reference to the example above. Hereafter a **discussion** will include the answering of the above questions together with pointing out any other peculiarities of the curve and the construction of the curve.

1. No, for $x = 0$, $y = 0$ cannot satisfy an equation having a constant term.

2. At $(0, -3)$ on the y-axis and at $(1, 0)$ and $(-3, 0)$ on the x-axis.

3. Yes. For as x increases indefinitely y also increases indefinitely.

4. No.

5. Yes, about a line parallel to the y-axis and through $(-1, 0)$.

6. No values of x are excluded. All values of $y < -4$ are to be excluded. For such values of y, x is imaginary, as can be seen by substituting in the equation.

What are the roots of the equation $x^2 + 2x - 3 = 0$? Compare these values with the x-intercepts.

1. Discuss $y = x^2 - 10x + 5$.
2. " $y = 12x^2 + 3x$.

105. Implicit quadratic functions. — Three important cases will be considered:

(*a*) $$9x^2 - 16y^2 - 144 = 0.$$

Solving for y,

$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 16}.$$

$x =$	$-8,$	$-6,$	$-5,$	$-4,$	$-2,$	$0.$
$y =$	$\pm 5.2,$	$\pm 3.1,$	$\pm 2.2,$	$\pm 0,$	$\pm i,$	$\pm i.$
$x =$	$2,$	$4,$	$5,$	$6,$	$8.$	
$y =$	$\pm i,$	$\pm 0,$	$\pm 2.2,$	$\pm 3.1,$	$\pm 5.2.$	

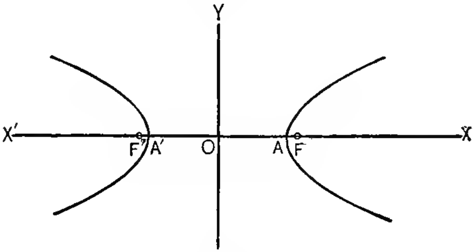


FIG. 84.

Here i is used to denote that the value is imaginary. The graph is given in the figure. Let the student discuss fully. This curve is called a hyperbola.

(*b*) $$xy = 12.$$

$x =$	$\frac{1}{4},$	$\frac{1}{2},$	$1,$	$2,$	$3,$	$4,$	$6,$	$12.$
$y =$	$48,$	$24,$	$12,$	$6,$	$4,$	$3,$	$2,$	$1.$
$x =$	$-\frac{1}{4},$	$-\frac{1}{2},$	$-1,$	$-2,$	$-3,$	$-4,$	$-6,$	$-12,$
$y =$	$-48,$	$-24,$	$-12,$	$-6,$	$-4,$	$-3,$	$-2,$	$-1.$

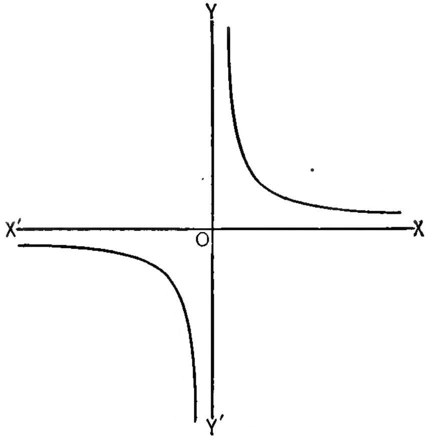


Fig. 85.

The graph is given in the figure. Let the student discuss fully. This curve is also a hyperbola:

(c) $$9\,x^2 + 16\,y^2 = 144.$$

$x = -5,\ -4,\ \ -3,\ \ -2,\ -1,\ \ 0,\ \ 1,\ \ \ 2,\ \ 3,\ 4,\ \ 5.$
$y = \pm i,\ \pm 0,\ \pm 1.9,\ \pm 2.6,\ \pm 3,\ \pm 3,\ \pm 3,\ \pm 2.6,\ 1.9,\ 0,\ \pm i.$

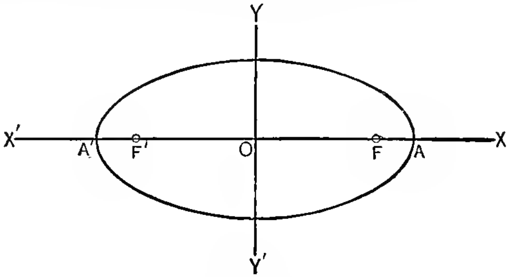


Fig. 86.

The graph is shown in the figure. Let the student discuss fully. This curve is called an ellipse.

106. The curves of **104**, **105**, **34**, are curves of the second degree. They are called **conic sections**. The student should learn to recognize these curves and their equations.

107. Functions of the third degree will be illustrated by

$$y = 2x^3 - 5x^2 + x + 2.$$

$$\begin{array}{llllllllll} x = & -1, & -\frac{1}{2}, & 0, & \frac{1}{2}, & 1, & \frac{3}{2}, & 2, & 3. \\ y = & -3, & 0, & 1, & \frac{3}{4}, & 0, & -\frac{1}{2}, & 0, & 7. \end{array}$$

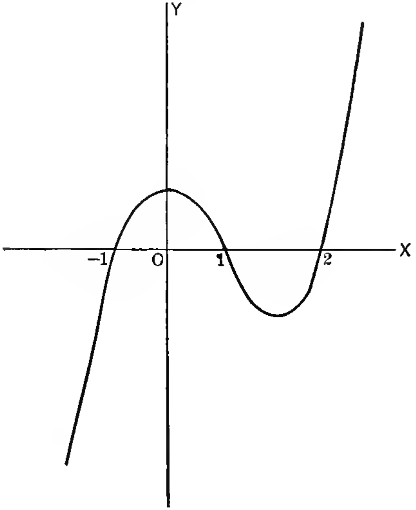


FIG. 87.

The graph is given in the figure. It is typical of all explicit cubic functions. Let the student discuss fully.

108. Rational fractional function. — Consider the equation

$$y = \frac{1}{x^3 - 3x^2 + 2x} = \frac{1}{x(x-1)(x-2)}.$$

The second form is more convenient for calculating:

$$\begin{array}{llllllllllllll} x = & -\infty, & -2, & -1, & -\frac{1}{2}, & 0, & \frac{1}{4}, & \frac{1}{2}, & 1, & \frac{3}{2}, & 2, & 3, & 4, & \infty. \\ y = & 0, & -\frac{1}{24}, & -\frac{1}{6}, & -\frac{8}{15}, & \mp\infty, & 3\frac{1}{21}, & \frac{3}{8}, & \pm\infty, & -\frac{8}{3}, & \mp\infty, & \frac{1}{6}, & \frac{1}{24}, & 0. \end{array}$$

The graph is shown in the figure. It should be noted that the infinite values of y occur at the zero values of the denominator. At these values the definition of continuity does not hold. The function is said to be discontinuous at such points. The value of x which gives an infinite value of y is called an infinity of the

function or a pole of the function. Let the student discuss the example fully.

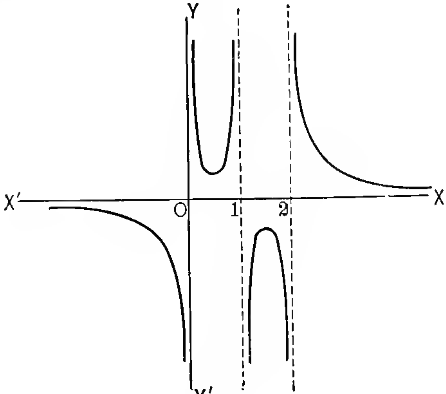


FIG. 88.

109. As an example of **irrational functions,** consider

$$y = \sqrt{x^3 - 5x^2 + 6x} = \sqrt{x(x-2)(x-3)}.$$

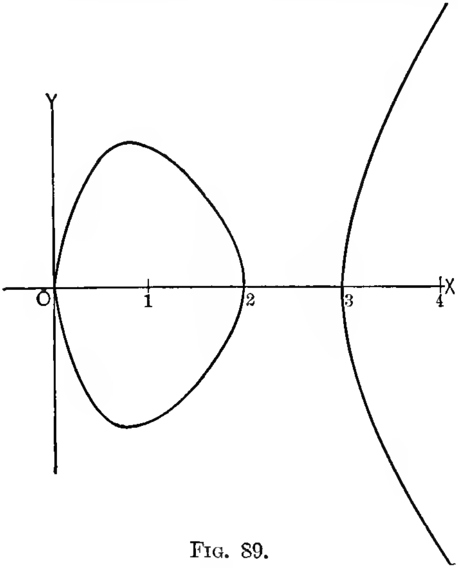


FIG. 89.

The second form is most convenient for calculating:

$$x = 0,\quad 1,\quad \tfrac{3}{2},\quad 2,\quad \tfrac{5}{2},\quad 3,\quad 4,\quad 5.$$
$$y = 0,\ \pm\sqrt{2},\ \pm\sqrt{\tfrac{9}{8}},\ \pm 0,\ \pm i,\ \pm 0,\ \pm\sqrt{8},\ \pm\sqrt{30}.$$

The curve is shown in the figure. Let the student discuss fully.

Construct the graphs and discuss fully the following:

1. $y = x^2 - x^{-1} + 5.$
2. $xy + y^2 = 23.$
3. $y = x^{\frac{3}{2}}.$
4. $y = \sqrt{2x^3 + x^2 - 2x + 2}.$
5. $(1 - x^2)\,y = x + 3$
6. $y = \sqrt{\dfrac{1-x}{1+x}}.$
7. $y^2 = \dfrac{x^3}{4 - x}.$
8. $xy + y^2 = 0.$
9. $x - 1 = \dfrac{1}{y - 1}.$
10. $y = (x - 3)^3.$
11. $y = \sqrt{\dfrac{x}{x+1}}.$
12. $x^2 + 4y^2 = 16.$

110. Simultaneous equations of the second and higher degrees in two unknowns can be solved (at least approximately) by means of their graphs. To do this construct the graphs of both equations on the same axes and to the same scale. The measured coördinates of the points of intersection of the graphs will be the solution of the pair of equations. The slide rule and tables of squares and cubes should be employed to facilitate calculation.

1. $y + x^2 = 7$ and $y^2 + x = 11$, find x and y.
2. $x^2 + y^2 = 25$ and $x + y + 1 = 0$, find x and y.
3. $x^2 + y^2 = 25$ and $x^2/9 + y^2/36 = 1$, find x and y.
4. $x^2/9 + y^2/36 = 1$ and $x^2/4 - y^2/16 = 1$, find x and y.
5. $y = x^3$ and $x^2 + y^2 = 25$, find x and y.
6. $x^2 + y^2 = 9$ and $y = \sin x$, find x and y.
7. $y = \cos x$ and $y = \sin x$, find x and y.

111. The question of **equivalence of equations** and systems of equations will not be discussed systematically. A few examples illustrating the meaning of the term and impressing the need of care in checking of results will be given.

Two equations in the same unknown are equivalent when every root of each is a root of the other. Thus

$$x^2 - 3x - 4 = 0 \quad \text{and} \quad 5x^2 - 15x - 20 = 0$$

are equivalent. Let the student solve and verify the statement. The equations

$$x^2 - 3x - 4 = 0 \quad \text{and} \quad x^2 - x + 2 = 0$$

are not equivalent. Let student solve and verify. The following operations on an equation lead in general to an equivalent equation.

1. Multiplication or division of both members by the same known number.

2. Addition or subtraction of the same expression on both sides of the equation.

3. Clearing of fractions in most ordinary cases.

The following operations may not lead to an equivalent equation.

4. Multiplication or division of both members by an expression containing the unknown (except as noted in (3)).

5. Clearing an equation of radicals.

1. Solve $\sqrt{x+5} - \sqrt{x-4} = 9$; then solve $\sqrt{x+5} + \sqrt{x-4} = 9$ and test results by substitution in the original equations.

2. Are $x - 7 - \sqrt{x-5} = 0$ and $x^2 - 15x + 54 = 0$ equivalent?

A system of simultaneous equations is equivalent to another system if every solution of each system is a solution of the other system. In solving systems of equations it is best to substitute all results in the original equations.

1. Show, by solving and substituting, that the system

$$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} x - y = 1 \\ x + y + 1 = 4 \end{cases}$$

2. Determine whether the systems

$$\begin{cases} 3x - 2y + 5 = 0 \\ x - y - 1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 3x - 2y + 4 = 0 \\ 9x - 6y + 12 = 0 \end{cases}$$

are equivalent.

112. Some cases of systems of quadratic equations in two unknowns are easily solved. A few commonly occurring cases will be given below:

(*a*) One equation linear and one quadratic. This case was given in Chap. I, **10.**

(*b*) Both equations of the second degree and containing only the squares of the unknowns. Thus

$$ax^2 + by^2 = c,$$
$$a'x^2 + b'y^2 = c'.$$

First, regard these equations as linear in which x^2, y^2, instead of x, y, are the unknowns. Solve for the values of x^2, y^2. Take the square root of each value of x^2, y^2 for the values of x, y.

(*c*) All terms containing unknowns are of the second degree but not necessarily the squares of the unknowns. Thus

$$x^2 + xy = 10,$$
$$y^2 - xy = 12.$$

Substitute $y = mx$ in both equations and get

$$x^2 + mx^2 = 10,$$
$$m^2x^2 - mx^2 = 12.$$

Solving each of the last equations for x^2 and equating results,

$$x^2 = \frac{10}{m+1} = \frac{12}{m^2 - m}.$$

Solving the last equation for m,

$$m = 2.652 \quad \text{and} \quad m = -0.452.$$

Whence by the equation $y = mx$, there is obtained

$$y = 2.652\,x \text{ and } y = -0.452\,x.$$

Substituting the values of m in $x^2 = \dfrac{10}{m+1}$ in succession,

$$x^2 = 2.738$$

whence $$x = \pm 1.652.$$

Substituting the value of x in $y = 2.652\,x$ gives

$$y = \pm 4.381.$$

Similarly using the other value of m will give other values of x, y. These should be tested by substituting in the original equations.

(*d*) To solve the system $\begin{cases} x^2 + y^2 = 25 \\ 6\,xy = 33 \end{cases}$

Divide the second equation by 3 and add to the first equation, obtaining

$$x^2 + 2\,xy + y^2 = 36.$$

Taking the square root

$$x + y = \pm 6 \text{ (two equations).}$$

By subtracting instead of adding as above there is obtained

$$x^2 - 2\,xy + y^2 = 14.$$

Taking the square root

$$x - y = \pm\sqrt{14} = \pm\, 3.74+ \text{ (two equations).}$$

Solving now these four equations of the first degree in pairs will give the desired solution. Test the results in the original equations.

(*e*) The system

$$\begin{cases} x^3 + y^3 = 8 \\ x + y \;= 2 \end{cases}$$

can be solved. First divide the first equation by the second, member by member,

$$x^2 - xy + y^2 = 4.$$

This equation together with the second equation form a system like the one described in (*a*). This system has, therefore, been treated in Chapter I. Complete the solution and test results.

1. Solve the system $\begin{cases} x^2 + y^2 = 25 \\ 4x^2 + 6y^2 = 24 \end{cases}$

2. Solve the system $\begin{cases} x^2 + y^2 = 36 \\ x^2 - y^2 = 1 \end{cases}$

3. Solve the system $\begin{cases} x^3 - y^3 = 24 \\ x - y = 4 \end{cases}$

4. Solve the system $\begin{cases} x^2 + xy = 3 \\ x^2 + y^2 = 5 \end{cases}$

5. Solve the system $\begin{cases} x^2 + y^2 = 49 \\ xy = \dfrac{25}{2} \end{cases}$

6. A piece of cloth on being wet shrinks 10 per cent in length and 6 per cent in width. The total loss of area is 5 sq. yd. How many square yards were in the piece originally?

7. If \$1000 at a certain rate for a certain time at simple interest amounts to \$1250 (principal and interest) and if the same principal for 3 years less time at 2 per cent lower rate amounts to \$1200, find the rate and the time.

8. A pole stands on a tower. A man 5′ high (to his eye) standing on level ground finds that at a certain distance from the foot of the tower the angle subtended (at his eye) by the tower is the same as that subtended by the pole. The tower is 50′ high and the pole 80′ high. Find the distance of the man from the foot of the tower.

9. A garden plot adjacent a wall is to be fenced. The area is to be 160 sq. yds. The length is to the breadth as 3, 2. Find the dimensions of the plot. (Fence on three sides.) Two solutions.

10. The sum of the squares of two numbers is 83. Their difference is 4. Find the numbers.

11. The area of a circular race track is 150,000 sq. ft. The inner diameter is to the outer diameter as 14 to 15. Find the inner and outer diameters of the track.

12. A rectangle has an area of 135 sq. rds. If lines are drawn from two opposite vertices to the diagonal joining the other vertices divide that diagonal into three equal parts. Find the dimensions of the rectangle.

CHAPTER XIII

TRANSFORMATION OF COÖRDINATES

113. It is of great advantage, in certain problems, to be able to simplify or to change the form of an equation. Certain troublesome terms may be removed or some other change may be made. One common way of attaining these results is to substitute for the variables certain linear functions of new variables. This is called a **linear transformation.** Such a transformation has the effect of moving the axes to a new position with reference to the curve whose equation is thus transformed without affecting the fundamental properties of the curve or the degree of the equation.

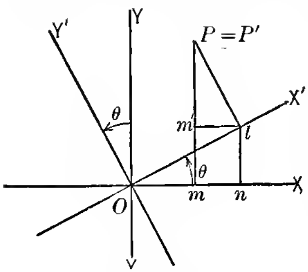


FIG. 90.

(*a*) To **rotate** the **axes** of coördinates through an assigned angle, θ, without moving the origin. The equations of transformation can be determined by considering only one point. For all points will be similarly affected by the transformation. Consider the point $P(x, y)$ referred to the axes OX, OY. Let the coördinates of the same point be x', y' referred to the new axes, OX', OY'. In the new position call $P = P'(x', y')$ which, of course, is the same point as $P(x, y)$ referred to the old axes.

From the figure $x = Om$, $y = mP$, $x' = Ol$, $y' = lP$. Hence

$$x = Om = On + nm.$$

But $\quad On = x' \cos\theta \quad$ and $\quad nm = y' \sin\theta.$

Therefore

(1) $$x = x' \cos\theta - y' \sin\theta.$$

In a similar manner,

(2) $$y = x' \sin\theta + y' \cos\theta.$$

Equations (1) and (2) are the ones desired. When θ is given x and y are expressed as linear functions of x' and y'. When θ is not known it may be found when certain other conditions are given.

(*b*) To **move** the **origin** without changing the direction of the axes.

It is easily seen from the figure that

(3) $$x = x' + h$$

and

(4) $$y = y' + k,$$

where the new origin is the point $O'(h, k)$, referred to the old axes.

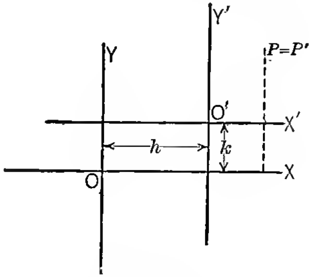


FIG. 91.

1. By use of (3), (4), move the origin to the point (2, 3) in the equation

$$x^2 + y^2 - 4x - 6y = 12.$$

Substituting $x = x' + 2$, $y = y' + 2$, into this equation,

$$(x' + 2)^2 + (y' + 3)^2 - 4(x' + 2) - 6(y' + 3) = 12.$$

Expanding and collecting, this reduces to

$$x'^2 + y'^2 = 1.$$

The primes may now be dropped if we remember that this equation is to be referred to the new axes. Hence the last equation may be written:

$$x^2 + y^2 = 1.$$

2. Using (1), (2) rotate the axes 45° in the positive direction in the equation $xy = 12$.

Substituting the values of x, y from (1), (2), for $\theta = 45°$, into this equation

$$(x' \cos 45° + y' \sin 45°)(x' \sin 45° - y' \cos 45°) = 12,$$

or

$$(0.707\,x' + 0.707\,y')\,(0.707\,x' - 0.707\,y') = 12,$$

or

$$0.5\,x'^2 - 0.5\,y'^2 = 12,$$

or dropping primes,

$$x^2 - y^2 = 24.$$

This equation represents the same curve referred to the new axes.

3. Determine h, k so that the first power terms in x and y shall disappear from the equation

$$x^2 - 4\,y^2 - 2\,x + 4\,y = 1.$$

Note. — After substituting from equations (3), (4), collect the coefficients of the first power of x and equate the result to 0. Similarly equate the coefficient of y to 0. The resulting two equations will determine h and k.

4. Determine θ so that the xy term shall disappear from

$$x^2 + xy + 2\,y^2 + x = 0.$$

Note. — Use Equations (1) and (2) and proceed as in the last exercise, putting the coefficient of xy equal to zero and solving for θ.

5. In $y = mx + b$, rotate the axes 90° in the positive direction. What is the meaning of m in the new equation? Of b?

6. In $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, rotate the axes 30°. Note the form of the resulting equation.

7. Using the first of equations (3) determine h so that the term in x^2 shall disappear. Draw graph of original and of new equation

$$y = x^3 + 3\,x^2 - 6\,x + 4.$$

8. By any method above remove the xy term from

$$x^2 - y^2 + 2\,xy + 3 = 0.$$

9. Change $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, to polar coördinates, using the transformation $x = r\cos\theta$, $y = r\sin\theta$ (**73**, Ex. 11, 12).

10. Change $y^2 = 8\,x$ to polar coördinates.

CHAPTER XIV

CONIC SECTIONS

114. The **conic sections** constitute the geometric aspect of integral functions of the second degree. In these are included all integral equations in two variables where at least one term is of the second degree. For this reason the conic sections are called loci of the second order, curves of the second order, or curves of the second degree.

Historically the geometric idea was developed first. The correlation of the geometric with the algebraic (analytic) idea with respect to these curves has been of much value in the study of the laws of nature.

Definition. — The locus of a point which moves in a plane so that the ratio of its distances from a **fixed point** and a **fixed straight line** is constant is called a conic section * or for short a conic.

This definition, based on the discovery of the property of these curves by the Greeks, furnishes a convenient starting point for an introductory analytic study of the curves.

The fixed point referred to in the definition is called the **focus** of the conic, and the fixed line the **directrix** of the conic.

The conics are conveniently classified, for purposes of elementary study, according to the **different values** which the **ratio**, referred to in the definition, may take. This ratio is called the eccentricity of the conic and will be denoted by e.

115. Ratio equal to one, $e = 1$. Parabola. — Let F be the focus, DD' the directrix and $P(x, y)$ any point on the curve.

* For proof that the curves of intersection of planes with a right circular cone have this property, see Wentworth Geom., Rev. Ed., p. 458, or some treatise on conic sections.

We wish to find the equation of this curve. Choose the origin on the curve midway between the focus and the directrix.

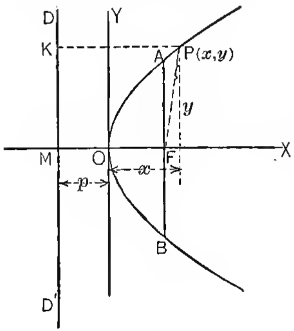


FIG. 92.

From the hypothesis and the figure we can write

$$FP = KP, \quad OF = MO = p.$$

Expressed in terms of x, y and p the relation

$$FP = KP$$

becomes

$$\sqrt{y^2 + (x - p)^2} = x + p$$

or (1) $$y^2 = 4\,px.$$

This is the desired equation of the parabola in the *standard form*. Let the student discuss the curve. This is the same kind of curve as the one given in **104**, the difference being the position with reference to the axes.

1. Find the total distance across the curve through the focus perpendicular to the x-axis. This double ordinate through the focus is called the **latus rectum.**

2. Move the origin to the point (h, k) and note the change in the equation.

3. Rotate the axes 90° in the negative direction and note the form of the resulting equation.

4. Find the equation of a parabola (standard form) that passes through $(5, 2)$. Determine the distance from the focus to the directrix and the distance from the focus to the origin.

5. Transform equation (1) so the origin will be at F.

6. Transform the resulting equation of example 5, to polar coördinates with pole at focus and polar axis the x-axis. Note form of equation.

7. Show that for any two points on a parabola, the squares of the ordinates are proportional to the abscissas (equation to be in standard form). Give a geometric interpretation of this theorem, that will be independent of the position of the axes or the form of the equation used.

8. Simplify the equation $y = 4x^2 - 6x + 40$, by putting it in the form $y = kx^2$.

Note. — The constant and the term in x may be removed by use of equation (3), (4), **113.**

9. Simplify and discuss $4y^2 - 6x + 3 = x - 5$.

10. Derive the equation of a parabola passing through (1, 1), (3, 5), (0, 0). Assume equation of form $(y - a)^2 = k(x - l)$ and determine a, k, l and write the equation accordingly.

116. Ratio less than one, $e < 1$. Ellipse. — Use the notation of **115** except that $MF = p = MA + AF$, $MA \neq AF$. Now from the figure and the hypothesis we can write

$$\begin{cases} \dfrac{AF}{MA} = \dfrac{FP}{KP} = e < 1 \\ AF = e \cdot MA. \end{cases}$$

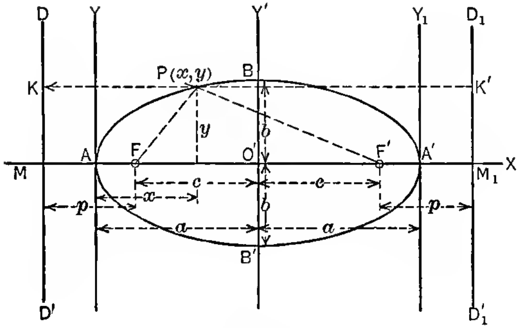


FIG. 93.

Solving for MA, AF,
$$MA = \frac{p}{1+e}$$
$$AF = \frac{ep}{1+e}.$$

Now $FP = e \cdot KP$ for all points of the curve. Expressing this relation in terms of the coördinates of $P(x, y)$ there is obtained

$$\sqrt{\left(x - \frac{ep}{1+e}\right)^2 + y^2} = e\left(\frac{p}{1+e} + x\right)$$

which reduces by clearing of radicals to

$$(1) \qquad (1 - e^2)\, x^2 - 2\, epx + y^2 = 0.$$

This is the equation of the ellipse in the position shown in the figure.

1. Discuss the curve from the above equation.

2. Move the origin to the point $\left(\frac{ep}{1 - e^2}, 0\right)$ to remove the term in the first power of x. (See **113** (b).)

3. Call $\frac{ep}{1 - e^2} = a$ and $(1 - e^2)\, a^2 = b^2$ in the equation obtained in example (2) and show that the equation reduces to

$$(2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the **standard form** of the equation of the ellipse. The origin is now O' and the y-axis is $O'Y'$ in the figure above.

4. Note the meaning of a, b in the figure and show that the intercepts on the x-axis are $-a$ and $+a$, and on the y-axis the intercepts are $-b$ and $+b$. The values a and b are the **semi-axes** of the ellipse. The value a is the **semimajor** axis and the value b is the **semiminor** axis; O' is the center.

5. Discuss the curve by use of the equation (2) above.

6. Call
$$\frac{ep}{1 - e^2} - \frac{ep}{1 - e} = \frac{e^2 p}{1 - e^2} = ae.$$
Solve this equation for p in terms of a and e. The result is
$$p = \frac{a\,(1 - e^2)}{e} = MF.$$

7. Find the value of MA in terms of a and e.

8. If $c = ae$, show by use of the values of a and b above that $a^2 - b^2 = c^2$ for any ellipse.

9. Show by use of the result of Ex. (8) that if $e = 0$ the ellipse becomes a circle.

Note. — This supposition makes $a = b$.

10. Find the equation of the locus of a point that is always 16 units from the point (3, 1).

11. A circle has its center at (3, 2) and passes through the point (8, 11). Find its equation.

Note. — Formulate the distance between (3, 2) and any point (x, y) of the curve and equate this expression to the given radius. Free the equation of radicals.

12. Find the equation of the ellipse, in standard form, if the eccentricity, e, is $\frac{2}{3}$ and the curve passes through (3, 4).

Note. — Form two equations and determine a and b of equation (2).

13. Find a, b, c, e and p in the ellipse $25x^2 + 144y^2 = 1500$.

14. Prove that for any ellipse $FP + F'P = 2a$ where P is any point on the curve.

15. What is the standard equation of

$$x^2 + 6y^2 - 6x + 8y = 4.$$

16. With the standard equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, move the origin to the left-hand focus $(-c, 0)$, then change to polar coördinates. Note the form. Compare with Ex. (6), **115.**

17. Find the length of the double ordinate through one focus in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and find the distance of the end of this ordinate from 0, and from the other focus. See Ex. 1, **115.**

117. Ratio greater than one, $e > 1$. Hyperbola. — With the same notation as in the last section and from the annexed figure may be written

$$(1) \qquad (e^2 - 1)x^2 + 2epx - y^2 = 0, \quad e > 1,$$

where the origin is at A and where $MA + AF = p$.

Since $(1 - e^2) < 0$, move the origin to $\left(\frac{-ep}{1 - e^2}, 0\right)$. The above equation becomes

$$(e^2 - 1)x^2 - y^2 = \frac{e^2p^2}{e^2 - 1}$$

or

$$x^2 - \frac{y^2}{1 - e^2} = \frac{e^2p^2}{(e^2 - 1)^2}.$$

Calling $\dfrac{ep}{e^2 - 1} = a$ and $a^2(e^2 - 1) = b^2$, the last equation may be put in the form

(2) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

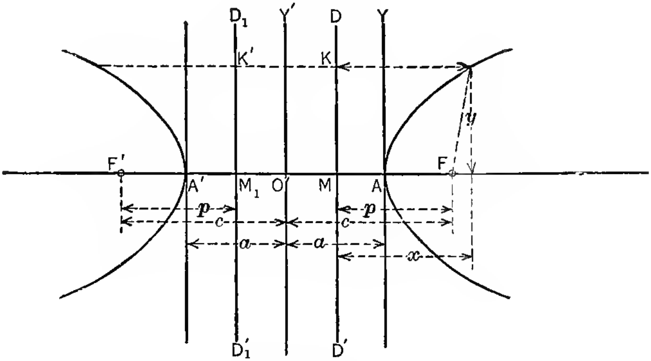


FIG. 94.

This is the **standard form** of the equation of the hyperbola referred to O' is the origin and $O'Y'$ the y-axis.

1. If $c = ae$, show that $a^2 + b^2 = c^2$.

2. Discuss the curve from equation (2).

3. Discuss the curve whose equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$, and compare with the curve of equation (2).

The curve of this exercise is called the **conjugate hyperbola** of the curve of equation (2). Note the intercepts of both curves on the axes.

4. Discuss the two curves $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ and $\dfrac{x^2}{16} - \dfrac{y^2}{9} = -1$. Draw the curves on the same axes.

5. Prove that for any hyperbola $F'P - FP = 2a$, equation (2).

6. With equation (2) rotate the axes 45° in the negative direction. Compare the result with the example of **105** (b).

7. With $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, move the origin to $(c, 0)$ and change to

polar coördinates. Note the form and compare with the polar forms of the parabola and the ellipse previously derived.

8. Find e, a, b, c, p for the curve $\frac{x^2}{25} - \frac{y^2}{16} = 1.$

9. Find the equation, in standard form, of a hyperbola of eccentricity, $e = 2$, and passing through $(7, 5)$. Find the equation of the conjugate hyperbola.

10. Find the equation of a hyperbola in standard form which passes through $(3, 5)$ and $(5, 7)$.

118. Diameter. — A line which bisects a system of **parallel chords** of any curve is a **diameter** of the curve.

All the conics have diameters. To illustrate, consider the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let P' (x', y') be the midpoint of any chord, AB. Let P (x, y) be a variable point on AB. Call $P'P = r$, and consider r positive if P is between P' and A, negative when P is between P' and B. Now from the figure write

$$\frac{x - x'}{r} = \cos\theta, \quad \frac{y - y'}{r} = \sin\theta.$$

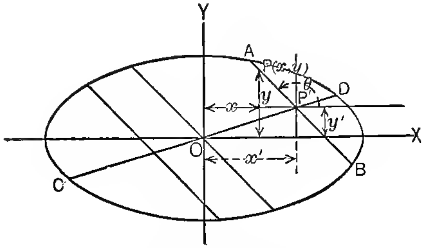


FIG. 95.

If P moves to A or to B its coördinates must satisfy the equation of the ellipse. Solving the last two equations for x, y and substituting in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives

$$\frac{x'^2 + 2\,rx'\cos\theta + r^2\cos^2\theta}{a^2} + \frac{y'^2 + 2\,ry'\sin\theta + r^2\sin^2\theta}{b^2} = 1.$$

When P is on the curve the roots of the last equation with regard to r are equal but of opposite signs. Hence the coefficient of the first power of r must be 0.* Therefore,

$$\frac{x' \cos\theta}{a^2} + \frac{y' \sin\theta}{b^2} = 0$$

or

$$y' = \frac{-b^2}{a^2} x' \cot\theta.$$

This is a linear relation between x', y', the coördinates of the midpoint of any (consequently all) chords making a given angle, θ, with the axis. The equation, therefore, holds for all points of the bisector of a parallel system of chords, and is the equation of the diameter which bisects these chords, that is of CD.

1. Find the equation of the diameter of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, which bisects the system of chords which make a 30° angle with the x-axis.

Note. — Adapt the method above to this case and work out in full.

2. Find by a method similar to the above, the equation of the diameter of the parabola which bisects all chords which make a 45° angle with the x-axis.

3. Prove that if the slope of a diameter of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is m and the slope of the corresponding system of chords is m', then $m \cdot m' = \frac{b^2}{a^2}$.

4. Show that if the diameter of an ellipse has a slope m, and the slope of the bisected chords is m', then the diameter which bisects the chords parallel to the first diameter has m' for its slope.

The two diameters referred to in this exercise are called **conjugate diameters**.

* See **92**, Ex. 5.

5. Find the equation of diameter of $\frac{x^2}{9} + \frac{y^2}{16} = 1$, which passes through the point $(1, 1)$ and find the equation of the diameter conjugate to this.

6. Find the equation of the diameter of $\frac{x^2}{16} - \frac{y^2}{9} = 1$, which bisects the chords which are parallel to $2x + 3y = 0$.

119. Eccentric angles. (*a*) **Ellipse.** — In the figure the large circle is called the major auxiliary circle, the small circle the minor auxiliary circle of the ellipse. P', P are points on the ellipse and major circle having the same abscissa. The angle, $\theta = AOP'$, is called the **eccentric angle** of the point, P, of the ellipse.

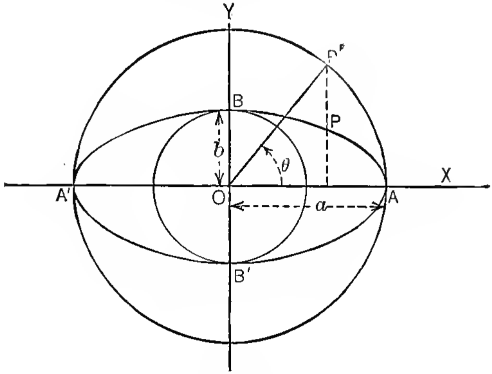


FIG. 96.

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x'^2 + y'^2 = a^2$, it is evident that for $x = x'$ the corresponding values of the y's are related by

$$y = \frac{b}{a} y'.$$

1. Suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be cut into narrow strips parallel to the y-axis (approximately rectangles). From the relation above show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

b/a times the area of the circle $x^2 + y^2 = a^2$. That is, that the area of the ellipse is πab.

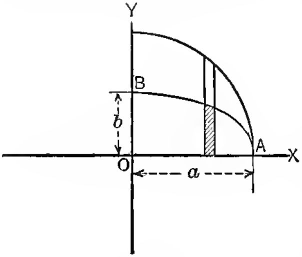


FIG. 97.

2. Show that the area of the ellipse is $\sqrt{1 - e^2}$ times the area of the circle given in example (1).

3. Show that if ϕ is the angle between the plane of the circle $x^2 + y^2 = a^2$ and the plane of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then $\sin\phi = e$ and $\cos\phi = \sqrt{1 - e^2}$ and that consequently from the results above $\cos\phi = b/a = \sqrt{1 - e^2}$. (Remember $e = c/a$, $c^2 = a^2 - b^2$.)

4. Find the equation of the ellipse which is the projection of the circle $x^2 + y^2 = 25$ on a plane at an angle $30°$ with the plane of the circle.

(*b*) **Hyperbola.** — In the figure P, P' are corresponding points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and its auxiliary circle $x^2 + y^2 = a^2$. The angle θ is called the **eccentric angle** of the point P, on the hyperbola. It is easily seen that

$$x = a\sec\theta, \quad y = b\tan\theta.$$

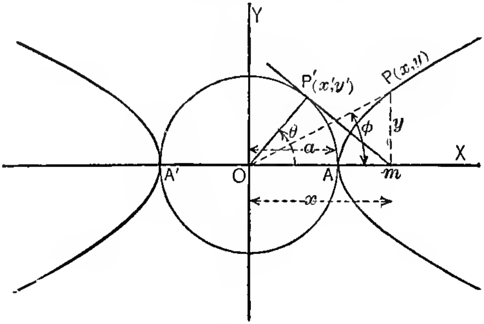


FIG. 98.

These relations are closely connected with a very interesting set of functions, known as the *hyperbolic functions*. These

functions have properties quite similar to those treated in Chapter VIII. (See McMahon Hyperbolic Functions.)

120. The **most general equation** of the second degree in two variables is of the form

(1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Under certain conditions the terms of the first degree may be removed by moving the origin. Assume

$$x = x' + h \quad \text{and} \quad y = y' + k$$

and substitute in (1); the result is

(2) $$Ax_1^2 + Bx'y' + Cy'^2 + (2\,Ah + Bk + D)\,x' + (Bh + 2\,Ck + E)\,y' + Ah^2 + Ck^2 + Dh + Ek + F = 0.$$

The terms of the first degree can be removed if and only if simultaneously the equations,

$$2\,Ah + Bk + D = 0,$$
$$Bh + 2\,Ck + E = 0,$$

hold for finite values of h and k. Solving for h and k gives

$$h = \frac{2\,CD - BD}{B^2 - 4\,AC} \quad \text{and} \quad k = \frac{2\,AE - BD}{B^2 - 4\,AC}.$$

These values are finite if $B^2 - 4\,AC = \Delta \neq 0$. Therefore, the first degree terms can be removed by moving the origin if $\Delta \neq 0$. The term in xy may always be removed by rotating the axes. It will then be sufficient to discuss in detail the equation,

(3) $$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

I. **Consider this equation** when $A \neq 0$, $C \neq 0$. Now $\Delta \neq 0$. The terms of first degree can be removed. The resulting equation will be of the form

$$Ax^2 + Cy^2 + F' = 0.$$

(*a*) If A and C are like signed and $F' \neq 0$, the equation represents an ellipse, real if the sign of F' is opposite that of A and C, imaginary if the sign of F' is the same as the sign of A and C. If F' is zero, the ellipse is a point ellipse.

(*b*) If A and C are unlike signed and $F' \neq 0$ the equation represents a hyperbola.

If $F' = 0$, the left member of (3) breaks up into two linear factors and represents two straight lines through the origin.

II. Consider the case where either A or C vanishes, say $A = 0$ and $C \neq 0$. Now equation (2) becomes

(4) $$Cy^2 + Dx + Ey + F = 0.$$

If $C = 0$ and $A \neq 0$, the equation reduces to

(5) $$Ax^2 + Dx + Ey + F = 0.$$

By moving the origin to a point $(h, 0)$ equation (4) may be reduced to

(6) $$Cy^2 + Dx = 0,$$

and by moving the origin to $(0, k)$ equation (5) may be reduced to

(7) $$Ax^2 + Ey = 0.$$

Equations (6), (7) represent parabolas.

In case the equations (4), (5) take the form

(8) $$Cy^2 + F' = 0,$$

(9) $$Ax^2 + F = 0,$$

respectively, they represent pairs of straight lines parallel to the axes.

121. Confocal conics. — Two conics are confocal when they have the same foci. Consider the two conics whose equations are:

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(2) $$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

The foci of these conics will coincide if $a^2 - b^2 = a_1^2 - b_1^2$, or $a^2 - a_1^2 = b^2 - b_1^2$. Let $a_1^2 = a^2 + \lambda$ and $b_1^2 = b^2 + \lambda$. Substituting in (2):

(3) $$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1.$$

The curve (3) is confocal with the curve (1) for all values of λ. Why?

1. Find the equation of a conic confocal with $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and passing through (5, 6).

Note. — Write $\frac{x^2}{16 + \lambda} + \frac{y^2}{9 + \lambda} = 1$. Substitute the coördinates of the given point in this equation and solve for λ. Having found λ, substitute its value in the above equation.

2. For what values of λ will the equation

$$\frac{x^2}{16 + \lambda} + \frac{y^2}{9 + \lambda} = 1$$

represent an ellipse? For what values of λ will it represent a hyperbola? See **116–117** and determine what values of λ make this equation like those referred to.

3. Find the equation of a conic confocal with $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and passing through (5, 7).

4. Determine whether $8x^2 + 12y^2 = 96$ and $3x^2 - 8y^2 = 24$ are confocal conics.

122. Centers of conics. — The ellipse and hyperbola have two perpendicular axes of symmetry and therefore have a center of symmetry. This point is called the center of the conic. The circle is considered as a special case of the ellipse. When the equations of the ellipse and hyperbola are in standard form the centers are at the origin. When the equations are not in standard form the center, referred to the original axes, is found by the values of h, k by which the terms of first degree are removed. The values of h and k being the coördinates of the center referred to the original coördinate axes. It is to be understood that the xy term has first been removed from the equation before applying this method.

Full treatment of the method of finding the centers of conics cannot be entered into in this course. Some examples will be of use in showing how to determine the center of a conic in certain cases.

(*a*) Find the center of the circle $x^2 + y^2 - 6x + 8y = 0$. This equation may be written in the form

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 25,$$

or

$$(x - 3)^2 + (y + 4)^2 = 25.$$

It is evident that if the origin be moved to the point $(3, -4)$ that the first degree terms will disappear and the equation would be in standard form. Therefore, the center of the circle is the point $(3, -4)$, referred to original axes of coördinates. Let the student draw the curve and verify this result.

(*b*) Find the center of $4x^2 + 6y^2 + 12x - 24y = 3$. This equation can be written in the form

$$4x^2 + 12x + 9 + 6y^2 - 24y + 24 = 36,$$

or

$$(2x + 3)^2 + 6(y - 2)^2 = 36.$$

It is evident if the origin be moved to $(-\frac{3}{2}, 2)$ the first power terms will vanish and the center will be at that point.

1. Solve each of the above problems by the method of moving the origin, **113** (*b*).

2. Show that center of the circle $x^2 + y^2 + Dx + Ey + F = 0$ is at the point $\left(\frac{-D}{2}, \frac{-E}{2}\right)$.

3. Show that the equation of the circle whose center is (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

4. Find the center of $x^2 - y^2 - 3y + 24 = 0$.

5. Find the center of $x^2 + y^2 - 8x + 16y - 20 = 0$.

MISCELLANEOUS PROBLEMS

1. Reduce $2x^2 - 5xy - 3y^2 + 9x - 13y + 10 = 0$ to one of the standard forms and determine a, b, c, e, p as the case may require.

2. What is the equation of the circle of radius 10 and center at the point (2, 3). (See Ex. 3, **122.**)

3. What is the length of the common chords of $x^2 + y^2 = 8$ and $9x^2 + 4y^2 = 36$? (The common chord joins two points of intersection.)

4. Find the equations of the straight lines which coincide with the common chords in Ex. 3.

5. Find the standard equation of the hyperbola whose foci are the points $(-3, 0)$ and $(3, 0)$ and eccentricity 1.5.

6. See if the curve $y^2 = 2x - \sqrt{7/6}$ touches the curve $3x^2 - 6y^2 = 1$.

Note. — Solve simultaneously and determine whether the curves cut or just touch each other.

7. Find the equation in standard form of an ellipse whose foci are $(-3, 0)$ and $(3, 0)$ and eccentricity $2/3$.

8. Find the equation of the rim of a 6″ stove pipe cut at an angle of 30° with the axis of the pipe. (Standard form.)

9. Find the equation of the boundary of the shadow of a circle of radius, r, on a plane making a 45° angle with the plane of the circle. (Standard form.) (Light vertically above plane.)

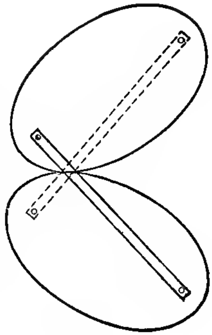
10. Find the equation of a circ e passing through the points $(1, 2)$, $(3, 5)$ and $(-1, 4)$ Find the lengths of the three chords joining these points.

Note. — Use $(x - a)^2 + (y - b)^2 = r^2$ and determine a, b and r.

11. Find the equation of a parabola, in standard form, which has its focus at $(4, 0)$.

12. If $x = vt$ and $y = -at^2$, eliminate t and discuss the locus of the resulting equation.

13. Prove that two elliptical gear wheels of the same size and shape will work together smoothly if connected by rods joining their foci as shown in the figure.



Ex. 13.

14. Determine the kind of conic represented by the following:

(*a*) $4x^2 + y^2 - 13x + 7y - 1 = 0$.
(*b*) $3x^2 - 4y^2 - 6y + 9 = 0$.
(*c*) $7x^2 - 17xy + 6y^2 + 23x - 2y - 20 = 0$.

15. Find a, b, c, e, p and the coördinates of the center of:

(*a*) $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$.
(*b*) $5x^2 + 6xy + 5y^2 + 22x - 6y + 21 = 0$.
(*c*) $5x^2 - 5xy - 7y^2 - 165x + 1320 = 0$.

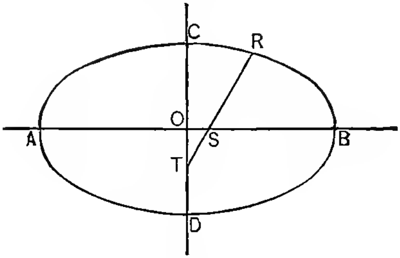
16. Find the equation of a conic confocal with $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and passing through the point (*a*), $(6, 5)$; (*b*), $(3, 2)$.

17. In the figure AB and CD are the axes of the ellipse. $RT = \frac{1}{2}AB$, $RS = \frac{1}{2}CD$. Show that if T be made to move on CD while S moves on AB, then R traces the ellipse. (This is the principle of the ellipsograph.)

18. By use of the property of the curve expressed in Ex. 14, **116**, devise a method of constructing the curve by the intersections of pairs of arcs whose centers are at the foci.

19. By use of the definition of the parabola, **115**, show how to construct

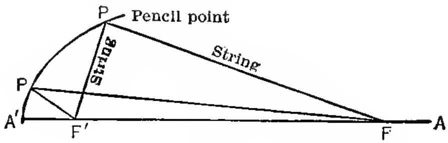
a parabola by intersection of arcs whose centers are at the focus with straight lines parallel to the directrix.



Ex. 17.

20. By use of the property of the curve expressed in Ex. 5, **117**, devise a method of constructing points on a hyperbola by intersections of arcs whose centers are at the foci.

21. Show that an ellipse can be drawn by using a string of length $2a$ fastening the ends of the string at the foci and holding a pencil against the string while drawing the curve. See figure below.



Ex. 21.

22. Show that the points (2, 2), $(-2, -2)$ and $(2\sqrt{3}, -2\sqrt{3})$ are the vertices of an equilateral triangle.

23. Prove that (10, 0), (5, 5), $(5, -5)$, $(-5, 5)$ are the vertices of a rapezoid.

24. In any triangle show that a line joining the middle points of two sides is parallel to the third side and equal to half of it.

Note. — Use two-point form and compare slopes.

25. Find the equation of a circle whose center is at (3, 2) and its radius 4.

26. Discuss the curve $y = x^2 - 2x - 3$.

27. Find the point of intersection of $x - 7y = 25$ with $x^2 + y^2 = 25$. Plot.

28. Change $x^2 + y^2 = 25$ to polar coördinates.

29. Move the origin so as to remove the first degree terms from $x^2 - y^2 - 6x + 4y - 12 = 0$ and discuss the curve with reference to the new axes.

CHAPTER XV

THEOREMS ON LIMITS, DERIVATIVES AND THEIR APPLICATIONS

123. Theorem I.* — The limit of the sum of two or more † infinitesimals, **37**, is zero.

Let δ_1, δ_2, δ_3 be three infinitesimals. By definition each must become and remain less than some arbitrarily small number, $\frac{\epsilon}{3}$ say, where ϵ is arbitrary. Hence at some stage and thereafter,

$$\delta_1 < \frac{\epsilon}{3}, \quad \delta_2 < \frac{\epsilon}{3}, \quad \delta_3 < \frac{\epsilon}{3}$$

and

$$\delta_1 + \delta_2 + \delta_3 < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Since ϵ is arbitrary the sum $\delta_1 + \delta_2 + \delta_3$ satisfies the definition of a limit and we conclude (see **37**)

$$\text{Lim } (\delta_1 + \delta_2 + \delta_3) = 0.$$

124. Theorem II. — The limit of the sum of two or more variables is the sum of their limits.

Let x, y, z be three variables and a, b, c their respective limits. Let

$$x - a = \delta_1, \quad y - b = \delta_2, \quad z - c = \delta_3.‡$$

Now from the definition of limit, δ_1, δ_2, δ_3, each has zero for its limit (see **37***d*). Write

$$x = a + \delta_1, \quad y = b + \delta_2, \quad z = c + \delta_2$$

* This and the three following theorems may be treated as assumptions if preferred and proofs omitted.

† In this and following theorems the term "more" is not to be interpreted to mean an infinite number.

‡ If $z < c$, δ_3 will be negative and similarly for the others.

and add

$$x + y + z = a + b + c + \delta_1 + \delta_2 + \delta_3.$$

The limit of the right side of this equation is $a + b + c$ (Theorem I). Therefore,

$$\text{Lim}\,(x + y + z) = \lim x + \lim y + \lim z.$$

125. Theorem III. — The **limit** of the **product** of two or more variables is the product of their limits.

With the same notation as in **124**, write

$$\begin{aligned}\text{Lim}\,xyz &= \text{Lim}\,(a + \delta_1)(b + \delta_2)(c + \delta_3)\\ &= \text{Lim}\,(abc + a\delta_2\delta_3 + b\delta_1\delta_3 + c\delta_1\delta_3 + ab\delta_3 + ac\delta_2\\ &\qquad + bc\delta_1 + \delta_1\delta_2\delta_3) = abc,\end{aligned}$$

since every term on the right after the first has zero for its limit. In a similar way the proof can be extended to any number of variables. The theorem is, therefore, true.

126. Theorem IV. — The **limit** of the **quotient** of one variable divided by another is the quotient of their respective limits.

With the same notation as above write

$$\text{Lim}\,\frac{x}{y} = \text{Lim}\,\frac{a - \delta_1}{b - \delta_2}.$$

By long division the right member may be written so that

$$\text{Lim}\,\frac{x}{y} = \text{Lim}\left(\frac{a}{b} + \frac{\frac{a}{b}\delta^2 - \delta_1}{b - \delta_2}\right) = \frac{a}{b},$$

since the numerator of the second fraction has zero for its limit. Therefore by **42**, III, the fraction has zero for its limit.

127. Definition and formation of the derivative of a function. — All numbers may be thought of as values which a variable may assume. Any number may be considered as the sum of two numbers, one a value which a variable may assume at some instant, the other an increment to the first so that the sum is a subsequent value of the variable. This way of thinking of numbers and in particular of symbols or variables which

assume number values yields a very useful instrument for solving certain kinds of problems which we have hitherto been unable to attack.

In order to make use of this idea it is necessary to learn to calculate and to manipulate a function called the derivative of another function.

The **derivative** of a function may be defined as the limit of the **ratio** of the **increment** of the **function** to the **increment** of the independent **variable** when the increment of the variable has **zero** for its limit.

In symbols this definition may be formulated as follows. Let the function be

$$y = f(x).$$

Then
$$\lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right),$$

if it has a limit, is the derivative of the function $f(x)$ with respect to x. The steps in calculating the derivative are:

$$y = f(x).$$

(1) $$y + \Delta y = f(x + \Delta x).$$

(2) Subtracting, $$\Delta y = f(x + \Delta x) - f(x).$$

(3) Dividing by Δx, $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

(4) Taking the limit,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The symbols $\frac{dy}{dx}$, $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, $f'(x)$, $D_x y$, $\frac{d}{dx} f(x)$ are to be used synonymously in this course as indicating the derivative of y with respect to x, where $y = f(x)$, or as the derivative of $f(x)$ with respect to x, where $y = f(x)$.

1. Calculate the derivative of $f(x) = x^3 + 2x + 1$ with respect to x. Write

$$y = x^3 + 2x + 1.$$

Giving x, y increments, Δx, Δy respectively

$$y + \Delta y = (x + \Delta x)^3 + 2(x + \Delta x) + 1$$
$$= x^3 + 3\,x^2\Delta x + 3\,x(\Delta x)^2 + (\Delta x)^3 + 2\,x + 2\,\Delta x + 1.$$

Subtracting, $\Delta y = 3\,x^2\Delta x + 3\,x\Delta x^2 + \Delta x^3 + 2\,\Delta x.$

Dividing by Δx, $\dfrac{\Delta y}{\Delta x} = 3\,x^2 + 3\,x\Delta x + (\Delta x)^3 + 2.$

Taking the limit, $\dfrac{dy}{dx} = 3\,x^2 + 2,$

since all other terms become zero when $\Delta x \to 0$, **124.**

2. Calculate the derivative of $f(x) = x - \dfrac{2\,x}{1+x}$, with respect to x. Write

$$y = x - \frac{2\,x}{1+x}.$$
$$y + \Delta y = x + \Delta x - \frac{2\,(x + \Delta x)}{1 + x + \Delta x}.$$
$$\Delta y = \Delta x - \frac{2\,\Delta x}{(1 + x + \Delta x)\,(1 + x)}$$
$$\frac{\Delta y}{\Delta x} = 1 - \frac{2}{(1 + x + \Delta x)\,(1 + x)}.$$
$$\frac{dy}{dx} = 1 - \frac{2}{(1+x)^2}.$$

3. Calculate the derivative with respect to x of

$$y = 3\,x^3 - 5\,x^2 + 2\,x - 1.$$

4. Calculate the derivative with respect to x of

$$y = 2 - x + 2\,x^2 - 3\,x^3.$$

5. Calculate the derivative with respect to x of

$$y = \frac{1}{1+x}.$$

6. Calculate the derivative with respect to x of

$$y = x - \frac{1}{x^2}.$$

7. Calculate the derivative with respect to x of

$$y = x^3 - \frac{4}{1 - x^2}.$$

8. Calculate the derivative with respect to x of

$$y = (x - 3)^4.$$

Note. — Expand first.

9. Calculate the derivative with respect to x of

$$y = x^2 + \frac{1}{x^2} + 6.$$

10. Calculate the derivative with respect to x of

$$y = 4x^2 - \frac{6}{1 + x}.$$

128. The above method of calculating derivatives is general but often difficult or laborious. To facilitate the calculation of derivatives, several special rules are in common use. These rules enable one to calculate with ease the derivatives of most ordinary functions.

129. The derivative of a constant is zero. — This follows from the fact that a constant can have but one value and therefore cannot admit an increment. That is, the increment of a constant is zero and consequently its derivative is zero. This may be symbolized as

$$\frac{d}{dx} C = 0, \tag{1}$$

where C is any constant and x any variable.

130. The derivative of x with respect to x is 1. For write

$$x = x.$$

Then

$$\Delta x = \Delta x,$$

$$\frac{\Delta x}{\Delta x} = 1,$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

$$\therefore \quad \frac{d}{dx} x = 1. \tag{1a}$$

131. Derivative of u^n, where u is a function of x and n a **positive integer.** — Write

$$y = u^n.$$

$$y + \Delta y = (u + \Delta u)^n = u^n + nu^{n-1}\Delta u + \frac{n(n-1)}{\lfloor 2} u^{n-2}\Delta u^2 *$$

$$+ \text{(terms containing } \Delta u^3 \text{ or higher powers)},$$

$$\Delta y = nu^{n-1}\Delta u + \frac{n(n-1)}{\lfloor 2} u^{n-2}\Delta u^2$$

$$+ \text{(terms containing } \Delta u^3 \text{ or higher powers)},$$

$$\frac{\Delta y}{\Delta x} = nu^{n-1}\frac{\Delta u}{\Delta x} + \frac{n(n-1)}{\lfloor 2} u^{n-2}\Delta u \cdot \frac{\Delta u}{\Delta x}$$

$$+ \text{(terms containing } \Delta u^2 \text{ or higher powers)}.$$

(2) $$\therefore \quad \frac{dy}{dx} = nu^{n-1}\frac{du}{dx},$$

since all other terms have zero for a limit as $\Delta x \rightarrow 0$, **124.**

This equation is true for all values of n, fractional, negative, irrational or imaginary. If desired the proofs in **132**, **133**, **134**, **135** may be omitted.

132. Derivative of u^n when n is a positive **fraction,** say $n = p/q$ where p and q are positive integers. Write

$$y = u^{p/q}.$$

$$y^q = u^p \quad \text{(Raising to } q\text{th power}$$

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx} \quad \text{(By } \mathbf{131}.)$$

$$\frac{dy}{dx} = p/q\,\frac{u^{p-1}}{y^{q-1}}\frac{du}{dx} = p/q\,\frac{u^{p-1}}{u^{p-p/q}}.$$

(3) $$\therefore \quad \frac{dy}{dx} = p/q\, u^{p/q-1}\frac{du}{dx}.$$

133. Derivative of u^n when n is negative. — Write

$$y = u^{-m} = \frac{1}{u^m}.$$

$$y + \Delta y = \frac{1}{(u + \Delta u)^m}.$$

* $\lfloor n = 1 \cdot 2 \cdot 3 \cdots n$, read factorial n.

$$\Delta y = \frac{1}{(u+\Delta u)^m} - \frac{1}{u^m}$$

$$= \frac{u^m - \left(u^m + mu^{m-1}\Delta u + \frac{m(m-1)}{\lfloor 2}u^{m-2}\Delta u^2 + \cdots + \Delta u^m\right)}{(u+\Delta u)^m u^m}$$

$$\frac{\Delta y}{\Delta x} = \frac{-\left(mu^{m-1}\frac{\Delta u}{\Delta x} + \frac{m(m-1)}{\lfloor 2}u^{m-2}\Delta u \cdot \frac{\Delta u}{\Delta x} + \cdots + \Delta u^{m-1}\frac{\Delta u}{\Delta x}\right)}{(u+\Delta u)^m u^m}.$$

(4) $$\therefore \quad \frac{dy}{dx} = -mu^{-m-1}\frac{du}{dx}.$$

134. Derivative of u^n when n is irrational. — Let m be a variable assuming only rational values as it approaches the irrational number n as a limit. Write $n = m + \epsilon$, where $\epsilon \rightarrow 0$, and consider

$$y = u^n = u^{m+\epsilon} = u^m \cdot u^\epsilon,$$

where ϵ is independent of x. Therefore, since $u^\epsilon \rightarrow 1$ independently of x we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\lim_{\epsilon\rightarrow 0} u^m u^\epsilon\right) = \frac{d}{dx}u^n.$$

(5) $$\therefore \quad \frac{dy}{dx} = nu^{n-1},$$

since $m \rightarrow n$ as $\epsilon \rightarrow 0$.

135. Derivative of u^n when n is imaginary. — Let $n = mi$, where $i^2 = -1$. Evidently if i is a number it is constant and so far as calculations are concerned offers no new principle. Hence we may write:

$$y = u^{mi}.$$

(6) $$\frac{dy}{dx} = mi\,u^{mi-1}\frac{du}{dx}.$$

Equations 2, 3, 4, 5, 6 show that the equation (2) can be used for all values of n.

136. Derivative of the product of two or more functions. — Write

$$y = uv,$$

where u and v are functions of x. Then

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$
$$= uv + u\Delta v + v\Delta u + \Delta u\,\Delta v.$$
$$\Delta y = u\Delta v + v\Delta u + \Delta u\,\Delta v.$$
$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta v \cdot \frac{\Delta u}{\Delta x}.$$

(7)
$$\frac{dy}{dy} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Exercise. — Apply formula (7) to show that the derivative of a constant times a function is the constant times the derivative of the function. That is,

(7*a*)
$$\frac{d}{dx}(Cu) = C\frac{du}{dx}.$$

137. Derivative of the quotient of one function divided by another. — Write

$$y = u/v.$$

Where u and v are functions of x:

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$
$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$
$$= \frac{v\Delta u - u\Delta v}{(v + \Delta v)\,v}.$$
$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v + \Delta v)\,v}.$$

(8)
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

138. Derivative obtained from an implicit function. — This method will be illustrated by examples. Consider

$$b^2x^2 + a^2y^2 = a^2\,b^2.$$

Apply the formulas of **129**, **130**, **131** to each term. The result is

$$2\,b^2x\frac{dx}{dx}+2\,a^2y\frac{dy}{dx}=0.$$

Solving for $\frac{dy}{dx}$,

(a) $$\frac{dy}{dx}=\frac{-b^2x}{a^2y},$$

since $\frac{dx}{dx}=1$. By use of the original equation y may be eliminated from this derivative, giving the result in terms of x.

Again consider

$$axy+b=0.$$

Applying the above formulas to each term,

$$ax\frac{dy}{dx}+ay\frac{dx}{dx}+0=0.$$

Hence

(b) $$\frac{dy}{dx}=-\frac{y}{x}.$$

Find the derivatives of the following:

1. $y=3\,x^4-6\,x+1$ (formulas (1) and (2)).
2. $y=2\,x(x^2+1)$ (as a product, $u=2\,x$, $v=(x^2+1)$ or write $2\,x^3+2\,x$.
3. $y=(x-2)\,(x^2+4\,x)$ (as a product $u=(x-2)$, $v=(x^2+4\,x)$ or multiply and use (1) and (2).
4. $y=\frac{x}{x+1}$ (formula (8)).
5. $y^2+2\,xy-x^2=10$ (implicit function **138**).
6. $y=\left(\frac{1+x}{2\,x+1}\right)^2$ (formulas (2) and (8)).
7. $y=x^3-6\,x+\frac{2}{x}-\frac{5}{x^6}.$
8. $y=(1-x^3)^{\frac{3}{2}}$, $u^{\frac{3}{2}}$ where $u=1-x^3$.
9. $x^2y^2+x^2+y^2+1=0.$

A derivative has, in general, a definite value for an assigned value of the variable. This value is found by substituting the value of the variable in the derivative.

10. $y^2 = 6x^3 + 4$, find $\dfrac{dy}{dx}$ for $x = 6$, and $x = 0$.

11. $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$ for x at the point $(3, 4)$.

12. $xy = 12$, find $\dfrac{dy}{dx}$ for $x = 12$. Find y from the equation.

13. $x/y + 1/x^2 = y/x + y$, find dy/dx for $x = 1$.

14. $y = (a^2 - x^2)^{\frac{1}{2}}$, find $\dfrac{dy}{dx}$ for $x = a$.

15. $y = \dfrac{1}{(1-x)^{\frac{1}{2}}}$, find $\dfrac{dy}{dx}$ for $x = 2$, and $x = 0$.

16. For what values of x is the derivative of $x^2 - x + 5$ equal to 0? Equal to 10? (Equate the derivative to 0 and solve for x. Similarly, for all values.)

17. For what values of x is the derivative $y = x^3 - 9x$ equal to 0? Equal to 4?

18. The area of a circle is $A = \pi r^2$. Find $\dfrac{dA}{dr}$ when $r = 12$.

19. From $xy = 12$, find dy/dx when $x = 1$, $x = 12$.

20. From $y^2 = 4px$, find dy/dx when $x = p$.

139. Use of the derivative to find the slope of a curve. — The slope of a straight line was defined in **28.** The slope of a curve at a point is the slope of the tangent line at that point.

Let

$$y = f(x) \tag{1}$$

be the equation of any curve and let $P_1\,(x_1, y_1)$ be any given point on the curve and $P\,(x, y)$ a variable point in the neighborhood of $P_1\,(x_1, y_1)$. The slope of the chord P_1P is

$$\frac{\Delta y_1}{\Delta x_1} = \frac{y - y_1}{x - x_1} = \tan\phi.$$

As P moves toward P_1 as a limiting position, the chord P_1P approaches the position of TP_1, the tangent at P_1. Then also

$$\lim_{\Delta x_1 \to 0} \frac{\Delta y_1}{\Delta x_1} = \lim_{x \to x_1} \frac{y - y_1}{x - x_1} = \left.\frac{dy}{dx}\right]_{x = x_1}^{*} = \tan\theta.$$

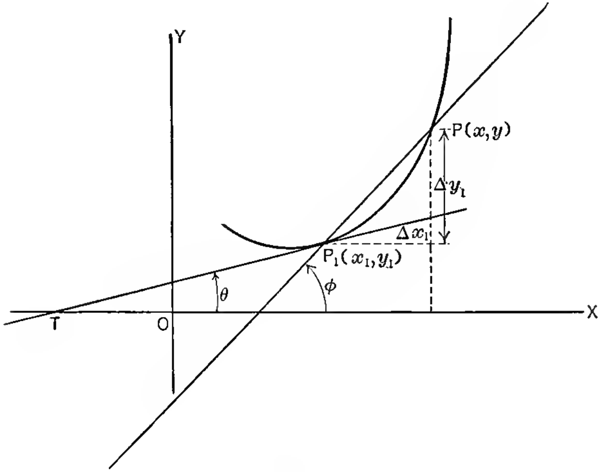


Fig. 99.

It follows that the slope of a curve at a given point and the slope of the tangent to the curve at that point are each equal to the derivative $\frac{dy}{dx}$ at that point.

It is now easy to obtain the equation of the tangent line to a curve at a given point, if the equation of the curve and the coördinates of the point are known. By **99**, Eq. 3, the equation of the line having a given slope and passing through a given point is

$$y - y_1 = m\,(x - x_1).$$

* The notation $\left.\frac{dy}{dx}\right]_{x = x_1}$ means that x_1 is substituted for x in the derivative, and y_1 for y also if y occurs in the derivative.

For a line tangent to $y = f(x)$ at the point (x_1, y_1) we must, therefore, have

$$(9) \qquad y - y_1 = \frac{dy}{dx}\bigg]_{x=x_1}(x - x_1).$$

This is the equation of the tangent line at (x_1, y_1).

1. Find the equation of the tangent to $y^2 = 8x$ at the point where $x = 8$.

Here
$$\frac{dy}{dx} = \frac{4}{y}.$$

By use of the equation of the curve, $y = 8$ when $x = 8$. Hence

$$\frac{dy}{dx}\bigg]_{x=8} = \frac{4}{y}\bigg]_{y=8} = \frac{1}{2}.$$

Then
$$y - 8 = \tfrac{1}{2}(x - 8)$$
or
$$2y - x = 8$$

is the desired equation of the tangent line.

2. Find the equation of the tangent to $y = \frac{x^2}{2}$ at the point where $x = 2$.

3. Find the equation of the tangent to $2y^2 - x^2 = 4$ at the point where $x = 4$.

4. Find the equation of the tangent to $x^2 + y^2 = 25$ at the point where $x = 3$.

5. Find the equation of the tangent to $y = x^3$ at the point where $x = -1$.

6. At what angle do $x^2 + y^2 = 25$ and $xy = 12$ intersect?

Note. — The angle between two curves at their point of intersection is the angle between their tangents at the point of intersection. See **102.**

7. At what angle do $x^2 - y^2 = 36$ and $2x - 3y = 1$ intersect?

8. Show that $y^2 = 4px$ has two tangents for $x = p$ and that these tangents meet each other at right angles on the x-axis.

9. Show that the tangent line at any point of the parabola $y^2 = 4px$ makes equal angles with the line from the focus to

the point of contact and a line through that point parallel to the x-axis. What practical use is made of this property?

10. Show that the tangent at any point of the ellipse makes equal angles with the lines from the foci to the point of contact.

140. Maximum and minimum values of functions are of frequent occurrence and of much importance. As a simple example, consider the case of a body thrown vertically upward, to find the greatest height it will ascend when its initial velocity is given. The law of the falling body must be combined with the law of uniform motion. The formula is

$$s = v_0 t - \tfrac{1}{2} g t^2,$$

where s is the height, v_0 the initial velocity, g the acceleration of gravity and t the time in seconds from starting.

Solving the equation for t gives

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 2\,gs}}{g}.$$

Since v_0, g, s are essentially positive and t must be real the expression $v_0^2 - 2\,gs$ must not be negative. Hence s may become so large as to make $v_0^2 - 2\,gs = 0$ but cannot increase further without making t imaginary. Hence solving $v_0^2 - 2\,gs = 0$ for s gives the maximum value of s to be $s = v_0^2/2\,g$. This method is not easy to carry out in most cases that arise in scientific investigations. A much more powerful and general method is given below.

141. Use of the derivative to determine maximum and minimum values of functions. — All ordinary functions of one variable admit of graphic representation. For this reason geometrical problems are convenient and sufficient to illustrate the use of the methods. These can be immediately transferred to any field of science by the medium of graphic representation which is of almost universal application.

Definitions. — A maximum ordinate of a curve is one which is algebraically greater than ordinates on both sides of itself however near these ordinates be chosen.

A minimum ordinate of a curve is one that is algebraically less than ordinates on both sides of itself, however near they are chosen.

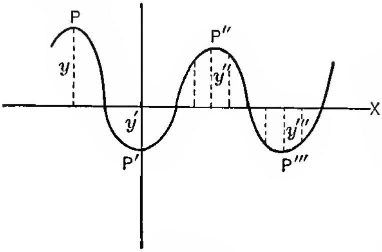


Fig. 100.

In Fig. 100, y and y'' are maximum ordinates and P and P'' are called maximum points of the curve. Similarly, y' and y''' are minimum ordinates.

Let $y = f(x)$ be any continuous function of x and suppose that its derivative is continuous and one-valued in the region considered. Then if $y_1 = f(x_1)$ is a maximum, we shall have

(1) $$f(x_1 - h) < f(x_1) > f(x_1 + h),$$

however small h is chosen. Similarly, for a minimum,

(2) $$f(x_1 - h) > f(x_1) < f(x_1 + h).$$

It follows from the continuity of the derivative and the definition of maximum and minimum, that if x_1 corresponds to a maximum of the function, the derivative $\frac{dy}{dx} = f'(x)$ will be positive for values of x in the interval $x_1 - h$ to x_1, negative in the interval x_1 to $x_1 + h$ and zero for $x = x_1$; this is easily seen from Fig. 101. For evidently the slope of the tangent is positive along the arc AP_1, negative along the arc P_1B and zero at P_1. Similarly if x_1 corresponds to a minimum the derivative $\frac{dy}{dx} = f'(x)$ will be negative for values of x in the interval $x_1 - h$ to x_1, positive in the interval x_1 to $x_1 + h$

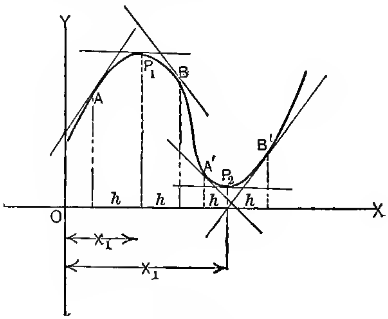


Fig. 101.

and zero for $x = x_1$. In the figure it is seen that the slope of the tangent is negative along the arc $A'P_2$, positive along the arc P_2B' and zero at P_2.

A necessary condition that $y = f(x)$ shall be a maximum or a minimum value at x_1 is that

$$\left.\frac{dy}{dx}\right]_{x=x_1} = f'(x_1) = 0.$$

It must be noted that this condition is not sufficient for it may hold at points where the function is neither a maximum nor a minimum as can be seen in the annexed figure.

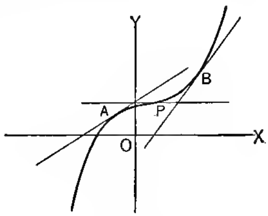


FIG. 102.

It is now necessary to determine when the condition above gives a maximum, a minimum or neither. This can be done by the use of the inequalities (1) and (2). The steps for determining maximum and minimum values of a function are:

1. Calculate the derivative with respect to the variable, and eliminate the dependent variable if it occurs.

2. Equate the derivative to zero, giving $f'(x) = 0$, if x is the variable.

3. Solve the equation in (2). Call its roots **critical values** of the variable. Denote them by $x_1, x_2, \ldots, x_n$.

4. To determine whether any one of these, say x_1, gives a maximum or a minimum, choose a convenient value of h and apply the inequalities (1), (2).

5. Having determined which critical values give maximum and which minimum values, substitute the critical values in the function and determine the actual maximum and minimum values of the function.

142. Consider $y = f(x)$, a function of x. If $f(x)$ increases in value when x increases and decreases when x decreases, $f(x)$ is called an **increasing** function of x. If $f(x)$ decreases when x increases and increases when x decreases, $f(x)$ is called a **decreasing** function of x.

Since, if $y = f(x)$ is a decreasing function of x, the increments Δy and Δx are of opposite signs, $\frac{dy}{dx} = f'(x)$ is negative. A given function may be an increasing function in one interval and a decreasing function in another interval.

Thus for a maximum:

(1) $$f'(x_1 - h) > f'(x_1) = 0 > f'(x_1 + h).$$

For a minimum:

(2) $$f'(x_1 - h) < f'(x_1) = 0 < f'(x_1 + h).$$

This method is quite similar in manipulation to the method explained in **141.** The student will find this method applicable to many problems when the second derivative is not easily obtained. (See **143** for second derivative.)

Consider $y^2 = 25\,x^2 - x^4$, and find the maximum value of this function. Solving for y,

$$f(x) = y = \pm\, x\sqrt{25 - x^2},$$

$$f'(x) = \frac{dy}{dx} = \frac{25 - 2\,x^2}{\sqrt{25 - x^2}} = 0 \text{ (use positive radical)},$$

$$x = \pm \frac{5}{\sqrt{2}} = \pm\, 3.54,$$

$$f'(x)\Big]_{x=3} = \frac{25 - 2 \cdot 3^2}{\sqrt{25 - 3^2}} = + \qquad (h = .54)$$

$$f'(x)\Big]_{x=4} = \frac{25 - 2 \cdot 4^2}{\sqrt{25 - 4^2}} = -. \qquad (h = .46)$$

Hence $f'(x)$ changes sign from $+$ to $-$ at the point $x = 3.54+$ and $f(x)$ is a maximum for this value of x. Substitute $x = 3.54$ in $f(x)$ and determine the maximum value.

Let the student test the negative value of $x = -3.54$. Also solve the problem using the negative radical. Draw a graph of the function and discuss this curve.

143. In general the derivative $f'(x)$ of $f(x)$ is itself a function of x and will have a derivative with respect to x. The derivative of $f'(x)$ is the second derivative of $f(x)$ and is denoted by the symbols $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}f(x)$ or $f''(x)$.

In the neighborhood of a maximum point the function $f(x')$ of $y = f(x)$ was shown to be first positive, then zero and negative for increasing values of x. It is, therefore, a decreasing function of x. Therefore if x_1 corresponds to a maximum value of $y = f(x)$ we must have

$$f'(x_1) = 0$$

and

$$f''(x_1) < 0.$$

At a minimum point we must have in a similar way

$$f'(x_1) = 0.$$
$$f''(x_1) > 0.$$

These inequalities and equations can be used for determining maximum and minimum values of functions instead of the methods of **141**, **142**.

Some illustrative examples will now be given.

1. Solve the problem at the beginning of **140** by use of the derivative. Write

$$s = v_0t - \tfrac{1}{2}gt^2.$$

Calculate the derivative of s with respect to t,

$$\frac{ds}{dt} = v_0 - gt.$$

For a maximum this derivative must be zero. Hence

$$v_0 - gt = 0,$$
$$t = v_0/g.$$

This is the critical value of t. Calculate the second derivative

$$\frac{d^2s}{dt^2} = -g.$$

This is negative for all values of t, since g is independent of t. The function, therefore, has a maximum for $t = v_0/g$. Substituting this value of t in the function gives

$$s = v_0 \cdot \frac{v_0}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0{}^2}{2g}$$

which is the same result as obtained before.

Instead of using the second derivative we may choose a value of h and try to satisfy inequality (1) **141.** Thus, let $h = \epsilon$, a small number, and substitute $t = \dfrac{v_0}{g} - \epsilon$ and $t = \dfrac{v_0}{g} + \epsilon$ in the original function and compare the result with that for

$$t = v_0/g.$$

For $t = \dfrac{v_0}{g} - \epsilon,\quad s = v_0\left(\dfrac{v_0}{g} - \epsilon\right) - \dfrac{1}{2}g\left(\dfrac{v_0}{g} - \epsilon\right)^2 = \dfrac{v_0^2}{2g} - \dfrac{1}{2}\epsilon^2 g.$

For $t = \dfrac{v_0}{g} + \epsilon,\quad s = v_0\left(\dfrac{v_0}{g} + \epsilon\right) - \dfrac{1}{2}g\left(\dfrac{v_0}{g} + \epsilon\right)^2 = \dfrac{v_0^2}{2g} - \dfrac{1}{2}\epsilon^2 g.$

For $t = \dfrac{v_0}{g},\quad s = \dfrac{v_0^2}{2g}.$

The last value is greater than the first two for all values of ϵ, however small. Then the last value of s is a maximum. This solution illustrates both methods of procedure.

2. Consider the problem: To prove that of all rectangles having a given area the square has a minimum perimeter. Let $A = xy$, where x, y are the length and breadth of the rectangle respectively. The perimeter is

$$P = 2x + 2y,$$

$$\frac{dP}{dx} = 2 + 2\frac{dy}{dx},$$

$$2 + 2\frac{dy}{dx} = 0.$$

To eliminate $\dfrac{dy}{dx}$ take the derivative of $A = xy$,

$$\frac{dA}{dx} = y + x\frac{dy}{dx} = 0,$$

since A is constant.

$$\therefore \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Substituting in $\dfrac{dP}{dx} = 0$ gives

$$2 - 2y/x = 0$$

or

$$x = y.$$

That is the length is equal to the breadth and the rectangle is a square. From $A = xy$ and $x = y$

$$x^2 = y^2 = A$$

and
$$x = y = \sqrt{A},$$

critical value of x. Choose $x = \sqrt{A} - h$, whence by $A = xy$,

$$y = \frac{A}{\sqrt{A} - h}.$$

Hence $$P = 2x + 2y = 2\left[\sqrt{A} - h + \frac{A}{\sqrt{A} - h}\right] > 4\sqrt{A}.$$

Choose $x = \sqrt{A} + h$, whence by $A = xy$,

$$y = \frac{A}{\sqrt{A} + h}.$$

Hence $$P = 2x + 2y = 2\left[\sqrt{A} + h + \frac{A}{\sqrt{A} + h}\right] > 4\sqrt{A}.$$

These results show that if $x \neq y$ the perimeter is greater than when $x = y$ and the minimum value of P is $4\sqrt{A}$.

To apply the method of **143** we have

$$\frac{d^2P}{dx^2} = 2\frac{d^2y}{dx^2}.$$

From $y = \dfrac{A}{x}$; $$\frac{d^2y}{dx^2} = \frac{2A}{x^3}.$$

Substituting the critical value, $x = +\sqrt{A}$, gives

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3}\bigg]_{x=\sqrt{A}} = \frac{4A}{\sqrt{A^3}} > 0, \text{ since } A > 0.$$

This test shows that the value $x = \sqrt{A}$ makes P a minimum and consequently from the condition above $x = y$, the rectangle is a square. Both methods lead to the same conclusion.

3. What are the dimensions of a tomato can of capacity 63 cu. in. of such form as to require a minimum of material, no allowance being made for seams?

Write

$$V = \pi r^2 h = 63, \tag{1}$$

where r is the radius and h the height of the can. Also,

(2) $$A = 2\pi r^2 + 2\pi rh,$$

where A is the area of the total surface of the can. Then

$$\frac{dA}{dr} = 4\pi r + 2\pi h + 2\pi r\frac{dh}{dr} = 0,$$

for a minimum. This becomes

(3) $$2r + h + r\frac{dh}{dr} = 0.$$

From (2),

(3a) $$\frac{dh}{dr} = -\frac{2h}{r}.$$

Substituting in (3) $\quad 2r + h - 2h = 0$

or $$2r = h.$$

From (1) by substitution $2\pi r^3 = 63.$

Whence $\quad r = 2.15$ (critical value of r)

and $\quad h = 4.30.$

From (3),

(4) $$\frac{d^2A}{dr^2} = 4\pi + 4\pi\frac{dh}{dr} + 2\pi r\frac{d^2h}{dr^2}.$$

From (3a), $$\frac{d^2h}{dr^2} = \frac{6h}{r^2}.$$

Substituting values of $\frac{dh}{dr}$, $\frac{d^2h}{dr^2}$, r and h in (4),

(5) $$\frac{d^2A}{dr^2} = 4\pi - \frac{8\pi h}{r} + 12\pi\frac{h}{r} > 0,$$

and a minimum of A is indicated.

Substituting values of r and h in (2) gives

$$A = 87.1 \text{ sq. in.}$$

as the minimum value of A.

By substituting $r = 2$, $r = 2.3$ for r in the expression for A, remembering $h = 2r$, gives for A, respectively, the values

$$A = 88$$

and $$A = 88 -,$$

both of which are greater than the value obtained above. Thus again the two methods agree in their results.

Remark. — Hereafter the student may solve a problem by one method. He may choose which method to use. In general if the second derivative is easily obtained that method will be best.

1. Show that of all rectangles of given perimeter the square has a maximum area.

2. Show that of all rectangles inscribed in a circle, the square has a maximum area.

Note. — Assume the radius of the circle equal to r. Consider the geometric properties of the figure and form an expression for the area of the rectangle using a variable dimension and solve the problem.

3. Show that of all triangles of a given base and perimeter the isoceles triangle has the maximum area.

4. Find the dimensions of a cylinder of maximum volume that can be inscribed in a cone of radius 10 and altitude 20.

5. Find the dimensions of a cone of volume 3000 cu. ft. that shall have a minimum curved surface.

6. A fireplace is 2′ deep and 4′ high. Find the length of the longest straight pole that can be pushed up the 1′ chimney.

7. Find the lowest point (minimum ordinate) of the curve $y = x^2 + x - 6$.

8. Does the curve $y = \dfrac{x+6}{x}$ have maximum or minimum ordinate?

9. A carpenter has 108 sq. ft. of lumber. Find the dimensions of the box of maximum capacity (lid included) that he can make, making no allowance for joining.

10. A rectangular sheet of iron is 12″ x 18″. Find the dimensions of the rectangular pan of maximum capacity that can be made by folding up the edges.

11. Find the legs of the right triangle of maximum area that can be constructed on a hypotenuse of 24″.

12. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 12.

13. In problem 5, **28**, use the three points corresponding to the prices \$12, \$18, \$24, of the profit curve and determine an equation of the form $y = ax^2 + bx + c$. From this equation determine the maximum profit and the price corresponding.

144. Use of the derivative to define motion. — We are familiar with "speed of a train" or "rate of walking" of a person. It is customary to say the speed of a moving body is the distance traveled divided by the time required to travel the distance. This is only an approximate expression of the idea. What is meant is — the number of units distance covered, divided by the number of units of time required, is the numerical measure of the average speed over the distance during the interval of time.

If speed is not uniform during the interval it will be more than the average speed during part of the interval and less than the average during a part of the interval. If, at a given instant of time, the speed of a body becomes uniform, then the distance Δs, passed over in the time Δt, is the instantaneous speed at the given instant. We shall write, therefore,

$$\textbf{Instantaneous speed} = \frac{\Delta s}{\Delta t},$$

under the above assumptions.

Consider the example of a train moving as follows:

The train	moves		40 mi.	during		1 hr.	
"	"	"	25 "	"	the first	$\frac{1}{2}$ hr.	
"	"	"	13 "	"	"	$\frac{1}{4}$ "	
"	"	"	5 "	"	"	5 min.	
"	"	"	1.1 "	"	"	1 "	

The average speed for each interval is as follows:

For	1 hr.		40 mi. per hr.
"	first	$\frac{1}{2}$ hr.	50 " "
"	"	$\frac{1}{4}$ "	52 " "
"	"	5 min.	60 " "
"	"	1 "	66 " "

Which of the above values of the speed is probably nearest the actual speed at the beginning of the hour?

Suppose the interval of time be decreased toward zero as a limit. The speed will also approach some value as a limit. The limiting value of the ratio $\frac{\Delta s}{\Delta t}$, as $\Delta t \to 0$, is the instantaneous speed at the beginning of the interval, Δt. But we know that if s is some function of t, the limit

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is the derivative of s with respect to t. That is, the instantaneous speed is the derivative of s with respect to t, when s is regarded as a function of t. It is noted that so far as the mathematician is concerned s and t are merely two variables related in some way and that he may with equal propriety regard dy/dx as the instantaneous speed of y with regard to x, or x-rate of y.

Linear **acceleration** is defined as the rate (speed) of change of speed. Therefore linear acceleration may be written

$$\text{acceleration} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2},$$

where s is regarded as a function of time, t.

1. A circular plate 6″ in diameter is expanding by heat so that its radius is increasing at the rate of 1″ per sec. At what rate is the area of the plate increasing?

Write

(1) $$A = \pi r^2.$$

Then $$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Substituting $r = 3$ and $\frac{dr}{dt} = 1$ from the problem,

$$\frac{dA}{dt} = 6\pi \text{ sq. in. per sec.}$$

2. The edge of a cube is 12″ and is increasing at the rate of 2″ per sec. At what rate is the volume increasing? The surface?

3. At what rate is the area of a rectangle increasing if its sides a, b are each increasing at the rate of c'' per sec.?

4. Water runs into a conical vessel at the rate of 3 cu. in. per sec. The diameter of the top is 12″, the altitude is 18″, vertex downward. At what rate is the depth of the water increasing when it is 8″ deep in the vessel?

Note. — Consider a cylinder of height unknown and radius equal to the radius of the vessel at 8″ from the vertex and of volume 3 cu. in.

5. A man 6′ tall walks along a path which passes under a light 12′ above the path. He walks at rate of 3 mi. per hr. from the light. At what rate is his shadow changing length when he is 25′ from a point under the light?

6. A body moves so that its distance from a fixed point is given by $s = t^3 + 6t^2 - 10x + 18$.

(*a*) What is the speed when $t = 4$? When $t = 0$?

(*b*) What is the distance when $t = 4$? When $t = 5$?

(*c*) What is the acceleration when $t = 10$? When $t = 1$?

(*d*) Discuss the graph of each of the functions a, b, c, using t as abscissa.

7. Construct on the same axes the graphs of $y = t^3$ and of the speed and the acceleration of this function.

8. Construct the graphs of $s = 16 \cdot t^2$ and of its speed and acceleration functions.

9. A particle moves in a path so that the coördinates of its position are $x = 1 - t + t^2$, $y = 1 + t + t^2$. Show that the path is a parabola when referred to x, y coördinates and that the speed of the particle is an increasing function of t.

10. The electrical resistance of platinum wire varies as its temperature $\theta°$ C., according to the law $R = R_0 (1 + a\theta + b\theta^2)^{-1}$. Calculate $\dfrac{dR}{d\theta}$ and interpret the meaning of this derivative.

11. A body moves on $y^2 = 12x$. At what rate is it moving parallel to the x-axis if it moves 50′ per sec. in the path when $x = 10$.

Note.— Calculate the component of speed parallel to the x-axis.

145. Equal roots of equations. — From **85** it is evident any integral function in one variable (or unknown) can be written in the form

$$f(x) = k(x - r_1)(x - r_2)(x - r_3) \dots (x - r_n),$$

where k is a constant and $r_1, r_2, \dots r_n$ are the roots of the equation,

$$(1) \qquad f(x) = 0.$$

If m of these roots are equal, equation (1) may be put in the form,

$$(2) \quad f(x) = k(x - r_1)^m (x - r_2)(x - r_3) \dots (x - r_{n-m}) = 0.$$

Call $\quad k(x - r_2)(x - r_3) \dots (x - r_{n-m}) = \phi(x).$

Then (2) may be written

$$(3) \qquad f(x) = (x - r_1)^m \phi(x) = 0.$$

Now

$$(4) \qquad f'(x) = m(x - r_1)^{m-1} \phi(x) + (x - r_1)^m \phi'(x) = 0.$$

The highest common factor of $f(x)$ and $f'(x)$ contains the factor $(x - r_1)^{m-1}$. The equation,

$$(5) \qquad (x - r_1)^{m-1} = 0,$$

contains r_1 as a root one less time than does $f(x) = 0$. It is easy to see that if there were no multiple * roots in $f(x) = 0$, there could be no highest common factor containing x.

If (3) should be of the form,

$$(6) \qquad f(x) = k(x - r_1)^m (x - r_2)^l \phi(x) = 0,$$

then corresponding to (5) is

$$(x - r_1)^{m-1}(x - r_2)^{l-1} = 0.$$

The reasoning applies for any number of multiple roots. Hence to determine whether an equation has multiple roots and to determine these roots when they exist, we may proceed as follows:

(*a*) Calculate the derivative of $f(x)$.

* When the same root occurs two or more times in an equation it is called a multiple root.

(*b*) Calculate the highest common factor of $f(x)$ and $f'(x)$ and call it $F(x)$.

(*c*) Solve the equation $F(x) = 0$.

(*d*) Each root of $F(x) = 0$ will occur in $f(x) = 0$ one more time than in $F(x) = 0$.

1. Determine the multiple roots of $x^6 - 2x^2 + 2x - 2 = 0$.

2. Determine the multiple roots of $4x^3 - 16x^2 + 52x - 3 = 0$.

3. Determine the multiple roots of $x^4 - 16 = 0$.

4. Determine k so that the roots of $x^2 - 6x + k = 0$ shall have its roots equal.

146. Momentum is defined as the product of mass and velocity. Velocity being a vector, momentum is a vector. If only the numerical value of velocity is considered, the product of speed and mass is the numerical value of momentum. In this sense consider

$$M = mv, \tag{1}$$

where m is mass and v is speed. Taking the derivative

$$\frac{dM}{dt} = m\frac{dv}{dt} = ma = f, \tag{2}$$

since mass multiplied by the magnitude of acceleration is the magnitude of force. Equation (2) shows that force is the time rate of change of momentum.

CHAPTER XVI

SERIES; TRANSCENDENTAL FUNCTIONS

147. A sequence * is called arithmetic if the differences between successive numbers of the sequence are equal throughout the sequence. In other words when every number of the sequence after the first may be found by adding the same number to the preceding number the sequence is called arithmetic. The number added to a number of an arithmetic sequence to obtain the succeeding number is called the common difference. Thus, the numbers

$$2, \quad 5, \quad 8, \quad 11, \quad 14, \quad 17$$

form an arithmetic sequence whose common difference is 3.

A sequence is called geometric if each number after the first in the sequence is obtained from the preceding number by multiplying by the same number. The multiplier is called the common ratio of the sequence. Thus, the numbers

$$2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64$$

form a geometric sequence whose common ratio is 2.

The law connecting successive numbers of a sequence may be simple or complicated. When this law can be expressed in the form of an equation it is called a recursion formula.

148. If $a_1, a_2, a_3, \ldots, a_n$ is a sequence of numbers of any kind, then

$$(1) \qquad a_1 + a_2 + a_3 + \cdots + a_n$$

is called a series. If n is infinite the series is called an infinite series. The numbers $a_1, a_2, a_3 \ldots$ are called the terms of the series.

* For definition of sequence see **37c**.

The reason for studying series lies in the fact that a number of the functions with which we have to do in mathematics and its applications are most naturally studied by means of the series which represent them. Many problems are of such a nature as to lead quite naturally to series in their solution. For purposes of this course we shall consider only one class of series, *viz.*, convergent series.

A convergent series may be defined as follows: If

$$(2) \qquad S_n = a_1 + a_2 + a_3 + \cdot \qquad + a_n$$

is the sum of the first n terms of the series

$$(3) \qquad S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

and if S_n approaches a definite limit as n increases indefinitely the series S is said to be convergent. This means that by taking the sum of a sufficient number of terms of a convergent series the difference between this sum and the limit of that sum may be made as small as one pleases. A series with a finite number of terms is always convergent if the terms are finite. In (3) $S = \lim\limits_{n \to \infty} S_n$ is sometimes called the sum of the series. S will also be used to denote the sum of a finite series.

149. Arithmetic series. — Let the series be

$$(1) \quad S = a + (a + d) + (a + 2\,d) + \cdots + (a + (n - 1)\,d),$$

where d is the common difference, a the first term and n the number of terms. Write the same series in reverse order

$$(2) \qquad S = (a + (n - 1)\,d) + (a + (n - 2)\,d) + \cdots + (a + 2\,d) + (a + d + a).$$

Add (1) to (2),

$$(3) \quad 2\,S = [a + (a + (n - 1)\,d)] + [a + (a + (n - 1)\,d)] + \cdots + [a + (a + (n - 1)\,d)].$$

$$(4) \quad \text{Call} \qquad \boldsymbol{a + (n - 1)\,d = l.}$$

Then on solving for S, and using (4), (3) becomes

$$(5) \qquad \boldsymbol{S = n\frac{a + l}{2}}.$$

Equations (4) and (5) are sufficient to solve all problems relating to arithmetic series. Of the five quantities a, d, n, l, S, three must be known to find two for the two equations (4) and (5) are sufficient to determine two unknowns.

1. How many terms of the series $2 + 7 + 12 + \cdots$ must be taken to obtain a sum of at least 100.

Here $a = 2$, $d = 5$, $S = 100$, to find n, and l if needed. From (4) and (5),

$$\frac{(2 + l)\,n}{2} = 100,$$

$$l = 2 + (n - 1)\,5.$$

Eliminating l and solving for n gives

$$n = \frac{1 \pm 63^{+}}{10} = -6.2 \text{ and } +6.4.$$

Since n must be a positive integer and S must not be less than 100, n must be taken equal to 7. Hence $l = 32$.

2. Interpolate 25 terms between -16 and 36 so as to form an arithmetic series.

Here $a = -16$, $l = 36$, $n = 25 + 2 = 27$, to find d, and S if needed. From (4),

$$36 = -16 + 26\,d.$$

By use of (5),

$$d = 2.$$

$$S = 270.$$

The series may be written as:

$$\begin{aligned}&-16 - 14 - 12 - 10 - 8 - 6 - 4 - 2 + 0 + 2 + 4\\ &\quad + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24\\ &\quad + 26 + 28 + 30 + 32 + 34 + 36.\end{aligned}$$

3. Find the sum of 13 terms of the series whose first three terms are, $4, 3\frac{1}{3}, 2\frac{2}{3}, \cdots$.

4. Find the fifteenth term of the series,

$$-\tfrac{1}{4} - \tfrac{1}{12} + \tfrac{1}{12} + \tfrac{1}{4} + \cdots.$$

5. If the sum of an arithmetic series is 500, the number of terms, 10, the first term, 0, find the common difference and the last term.

6. A hundred apples lie on the ground in a straight line, 4′ apart. A basket is 4′ from the end of the row in the same line. A boy starts at the basket and gathers the apples into the basket one at a time. How far must he walk?

7. A triangular frame is strung with parallel wires $\frac{1}{2}''$ apart. The first wire is the base of the triangle, the last $\frac{1}{2}''$ from the vertex. The base is 24″ and the altitude 30″. Find the total length of all the wires. Will the angles of the triangle have any effect on the result?

8. The equation of a straight line is $y = 2x + 4$. Ordinates 1 unit distance apart are erected, beginning at the y-axis, and ending at $x = 20$. Find the sum of all these ordinates. Find the mean ordinate.

9. Find the sum of all the odd integers from 1 to the nth odd integer in terms of n.

10. Find the sum of the first n even integers in terms of n.

11. Can an infinite arithmetic series be convergent? Why?

149a. Geometric series. — Let the series be

(1) $$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Then

(2) $$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

(3) Subtracting $$(r - 1)S = ar^n - a$$

and (4) $$S = \frac{ar^n - a}{r - 1}.$$

Call (5) $$l = ar^{n-1},$$

the last term. Then substituting

(6) $$S = \frac{rl - a}{r - 1}.$$

Equations (5) and (6) are sufficient to solve all problems relating to geometric series. Any three of the five quantities a, r, l, n and S being given the other two can be found, for equations (5), (6) are sufficient to determine two unknowns.

1. How many terms of $3 + 6 + 12 + \cdots$ must be taken to obtain a sum of at least 150?

Here $a = 3$, $r = 2$, $S = 150$, to find n, l.

By (5), (6),

$$l = 3 \cdot 2^{n-1},$$

$$S = \frac{2\,l - 3}{2 - 1} = 2\,l - 3 = 150.$$

Eliminating l and solving for n

$$2^n = 51,$$

$$n = \frac{\log 51}{\log 2} = 5.6.$$

Since n must be a positive integer we must take $n = 6$ in order that S shall not be less than 150.

2. Interpolate 6 terms between -8 and 16 so as to form a geometric series.

Here $a = -8$, $n = 6 + 2 = 8$, $l = 16$, to find r, and S if needed.

By (5), $$16 = -8\,r^7.$$
Whence $$r^7 = -2$$
and $$7 \log r = \log 2 \text{ (numerically)},$$
$$\log r = \frac{\log 2}{7} = 0.0430,$$
$$r = 1.104 \text{ (numerically)}.$$

But in this case r is negative and its value is $r = -1.104$. The series is, therefore,

$$S = -8 + 8.832 - 9.751 + 10.77 - 11.89 + 13.9 - 14.50 + 16.01.$$

The result shows that the value of r is only approximate and to get more accurate results a more accurate value of r must be used. It is noted here that when r is negative every other term of the series is negative. S was not needed.

3. Find the sum of $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$ to ten terms.

Note. — Find the tenth term and then calculate the sum. Use equations (5), (6).

4. Find the first term of a series if the sum is 500, number of terms 10 and last term 100.

5. If $a = 5$, $l = 400$, $n = 10$ in a certain series, find r and S.

6. If $a + b + c + \cdots$ is a geometric series, show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is a geometric series.

7. If $1 + 2 + 4 + 8 + 16 + \cdots$ form a geometric series, continue the series to ten terms and find the sum.

8. Show that $a + \sqrt{ab} + b$ form a geometric series if a and b are both positive or both negative.

9. Show that if $\log a + \log b + \log c + \cdots$ form an arithmetic series, $a + b + c + \cdots$ form a geometric series.

10. On a false balance a certain object weighs 9 lbs. on one pan and 16 lbs. on the other pan. If x is the true weight show that $9 + x + 12$ form a geometric series.

150. Special case of geometric series. — Let $n \to \infty$ and $|r| < 1$ in

$$S = \frac{a - ar^n}{1 - r}.$$

(Note that this equation is really equation (6)). Since $|r| < 1$,

$$\lim_{n \to \infty} r^n = 0.*$$

Hence the above equation will become

$$S = \frac{a}{1 - r}. \tag{1}$$

Therefore when in a geometric series $|r| < 1$ and $n = \infty$, the sum S_n has a definite limit as $n \to \infty$. This fact will be of much use in later work.

* The truth of this is evident when we consider any proper fraction, say $\frac{2}{3}$, and raise it to higher and higher powers. Thus

$$\frac{2}{3},\ \frac{4}{9} = \left(\frac{2}{3}\right)^2,\ \frac{8}{27} = \left(\frac{2}{3}\right)^3,\ \frac{16}{81} = \left(\frac{2}{3}\right)^4, \ldots,\ \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n, \ldots$$

It is seen that each successive power of $\frac{2}{3}$ is less than the preceding and that by taking n sufficiently large $(\frac{2}{3})^n$ may be made as small as we please. To determine n so that $(\frac{2}{3})^n < \frac{1}{10}$, write

$$n\,(\log 2 - \log 3) < \log \tfrac{1}{10} < -1.$$

Hence $n\,(0.1761) > 1$ and $n > 5.7$ Since n is an integer, take $n = 6$.

1. Find the limit of the sum of $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}$ as $n \to \infty$.

2. Find the value of 4.4747 . . . in the form of a common fraction.

Note. — Write

$$S = 4 + \frac{47}{100} + \frac{47}{10000} + \cdots$$

and sum the geometric series beginning with $\frac{47}{100}$. Add the result to 4.

3. Find the value of 11.911911911 . . . in the form of a common fraction.

4. Find the limit of

$$\frac{1}{1+h} + \frac{1}{(1+h)^2} + \frac{1}{(1+h)^3} + \cdots \quad h > 0.$$

Note. — Find the sum of the geometric series whose ratio is

$$\frac{1}{1+h}.$$

5. Find the sum of $12 + 9 + 6\frac{3}{4} + \cdots$ to an infinite number of terms.

6. If $|r| < \frac{1}{2}$, show that any term of a geometric series is greater than the sum of all the terms that follow.

Note. — Let a be the first term and r be the ratio, $r < 1$. Form the series and sum all terms after a given term, compare the result with the given term.

7. If the sum of ten terms of a geometric series is 244 times the sum of the first five terms, find the ratio.

151. Harmonic series. — If the series

(1) $$S = a_1 + a_2 + a_3 + \cdots$$

is a harmonic series, then

(2) $$S' = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

is an arithmetic series and conversely.

Suppose $S = a + x + b$ is a harmonic series in which x is to be determined when a and b are given. By (2),

$$\frac{1}{a} + \frac{1}{x} + \frac{1}{b}$$

is an arithmetic series. Therefore

$$\frac{1}{x} - \frac{1}{a} = \frac{1}{b} - \frac{1}{x},$$

whence

$$x = \frac{2ab}{a+b}.$$

This value of x is called the **harmonic mean** of a and b. Hence the series $a + x + b$ above becomes

$$S = a + \frac{2ab}{a+b} + b.$$

In problems relating to harmonic series it is often better to take the reciprocals of the terms and form an arithmetic series. Solve the problem corresponding to the original and then pass back to the harmonic series. There is no general formula for finding the sum of a harmonic series.

1. Find the harmonic mean of 3 and 7.

2. Form an equation of the third degree whose roots are in harmonic series, the smallest root being 3 and the next largest 4. See **85**.

3. In the equation of Ex. 2, substitute $1/y$ for x and determine the roots of the resulting equation. Do they form any kind of series that you know?

4. Show that the geometric mean of two numbers is the geometric mean of their arithmetic mean and harmonic mean.

Note. — Let a and b be the numbers. Form the means indicated and compare as required in the problem. Arithmetic mean of two numbers is half their sum. Geometric mean of two numbers is the square root of their product.

152. Convergence of series. — So far as this course is concerned a series must be convergent to be useful in solving

problems. Only a few standard series which are in common use for studying certain functions that are of great importance in the applications of elementary mathematics will be treated. Only convergent series can have finite and determinate sums.

Consider

$$y = F(x) = \frac{1}{1 - x}.$$

The graph should be drawn to show the curve on both sides of $x = 1$.

A discontinuity occurs at $x = 1$. The value of y is ∞ for $x = 1$. By long division

$$(1) \qquad \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots.$$

This is a geometric series having x for its ratio, and having an infinite number of terms. The sum of the series (1) is, by **150**, for $|x| < 1$,

$$S = 1 + x + x^2 + x^3 + \cdots$$
$$= \frac{1}{1 - x},$$

which is precisely the function itself. This shows that (1) is an identity for values of $|x| < 1$. When $x = 1$ both sides of (1) become infinite. When $x = -1$ the left member is $\frac{1}{2}$ while the right member is 0 or 1 according as n is even or odd respectively. For $x > 1$ the left member is finite while the right member becomes infinite. The series on the right in (1) can represent the function on the left only when $|x| < 1$. For such values of x a finite number of terms will give an approximate value of the function.

For values of $|x| > 1$, a different series can be written which will represent the function. Thus

$$(2) \quad \frac{1}{1-x} = -\frac{1}{x}\left(\frac{1}{1 - 1/x}\right) = -\frac{1}{x}\left(1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n} + \cdots\right).$$

The last series is a geometric series whose ratio is $\frac{1}{x}$ and therefore has a definite sum for values of $|x| > 1$. A finite number of

terms of the series in (2) will give an approximate value of the function. The series (2) does not converge if $|x| \geqq 1$ and therefore for such values of x cannot represent the function

$$\frac{1}{1-x}.$$

We shall study functions which can be represented as functions of the variable only in the form of series.

153. To use series in the study of functions it is necessary to be able to determine in a given case whether the series is convergent. The two following simple tests will meet present needs.

(*a*) *Comparison test.* — Suppose

(1) $$S = u_1 + u_2 + u_3 \cdots + u_n + u_{n+1} + \cdots$$

is an infinite series and let

(2) $$S' = v_1 + v_2 + v_3 + \cdots + v_n + v_{n+1} + \cdots$$

be an infinite series known to be convergent. If, beginning at any term of S, the terms of S' and the remaining terms of S can be paired off in such a way that every term of S is less in absolute value than its mate in S', or at most equal to it, the series S is convergent. For suppose

$$|u_k| < |v_1|, \quad |u_{k+1}| < |v_2|, \; \ldots, \; |u_{k+p}| < |v_{p+1}|, \quad \ldots$$

for all values of p, where k is finite. It is evident on adding these inequalities,

$$u_k + u_{k+1} + \cdots + u_{k+p} + \cdots < v_1 + v_2 + \cdots + v_p + v_{p+1} + \cdots$$

Since S' is convergent it follows that S must be convergent.

154. To employ the comparison test it is necessary to have several standard convergent series for comparison with unknown series. Let

(1) $$S_g = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots,$$

where $|r| < 1$. This series was shown in **150** to be convergent. Let

(2) $$S_p = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots.$$

If $p > 1$, S_p is convergent. For, write

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{2}{2^p}=\frac{1}{2^{p-1}};\quad \frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}=\frac{1}{4^{p-1}};\quad \frac{1}{8^p}+\frac{1}{9^p}+\cdots+\frac{1}{15^p}<\frac{8}{8^p}=\frac{1}{8^{p-1}};\ \cdots$$

Adding these inequalities,

$$S_p<\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\cdots.$$

This is a geometric series whose ratio is $\frac{1}{2^{p-1}}$ and is convergent if

$$\left|\frac{1}{2^{p-1}}\right|<1,$$

that is, if $p > 1$. If $p = 1$, $\frac{1}{2^{p-1}} = 1$ and the series does not converge. Series (2) is known as the p-series.

Let

(3) $$S = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots.$$

Now

$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{2};\quad \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{1}{2}\cdots.$$

Therefore

$$S>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\cdots.$$

By taking n sufficiently large S_n may become larger than any pre-assigned number M. The series does not converge. This series is called the harmonic series.

By comparing a given series with one or another of (1), (2), (3), the convergence or non-convergence of a number of series can be determined.

1. Prove that if $S = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ does not converge, and if $S' = v_1 + v_2 + v_3 + \cdots + v_n + \cdots$ is such that $v_1 > u_1$; $v_2 > u_2$; $v_3 > u_3$; $\ldots$; $v_n > u_n + \cdots$ the series S' does not converge. Consider positive values only.

2. Determine the convergence of:

$a.$ $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ (use p-series, $p > 1$).

$b.$ $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4}$ (compare with a geometric series).

$c.$ $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$ (use p-series, $p < 1$).

155. Ratio test for convergence. — Let

$$S = u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots. \tag{1}$$

If $\left| \lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n} \right) \right| = k < 1$, where k is a fixed number, the series (1) is convergent.

Write

$$\begin{array}{lcl} \frac{u_2}{u_1} \lesseqqgtr r_1 & \text{or} & u_2 \lesseqqgtr u_1 r_1. \\ \frac{u_3}{u_2} \lesseqqgtr r_2 & \text{or} & u_3 \lesseqqgtr u_1 r_2. \\ \vdots & \vdots & \vdots \\ \frac{u_{n+1}}{u_n} \lesseqqgtr r_n & \text{or} & u_{n+1} \lesseqqgtr u_n r_n. \\ \vdots & & \vdots \end{array}$$

Now if a common value can be assigned to the r's so that the above inequalities still hold we may add and obtain the result,

$$S = u_1 + u_2 + \cdots + u_n + u_{n+1} + \cdots \lesseqqgtr u_1 + u_1 r + u_1 r^2 + \cdots + u_1 r^n + u_1 r^{n+1} + \cdots.$$

If, further, $|r| < 1$ the series is convergent. For the right side of the last equation then becomes a geometric series whose ratio is less than one. If $|r| > 1$ the terms of S increase with n and the series does not converge. If $|r| = 1$, the test does not give reliable results, and some other test must be employed.

1. Test for convergence

$$S = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots.$$

Note. — The nth term is $\dfrac{1}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}$.

2. Test for convergence

$$S = 1 - \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots.$$

3. Test for convergence the series of Ex. 2, **154.**

156. Series with complex terms. — The real parts of all terms may be separated from the imaginary parts and arranged in a series. The imaginary parts may also be arranged in a series by themselves. Consider

$$S = (a_1 + b_1 i) + (a_2 + b_2 i) + \cdots + (a_n + b_n i) + \cdots.$$

Call $\qquad X = a_1 + a_2 + \cdots + a_n + \cdots$

and $\qquad Y = b_1 + b_2 + \cdots + b_n + \cdots.$

Since $|S| = \sqrt{X^2 + Y^2}$, evidently S is finite if both X and Y are finite. But X is finite if the series $a_1 + a_2 + \cdots + a_n + \cdots$ is convergent. Similarly, Y is finite if the series $b_1 + b_2 + \cdots + b_n + \cdots$ is convergent. Hence the convergence of the series of complex terms follows. The diagram illustrates the nature of a series of complex terms.

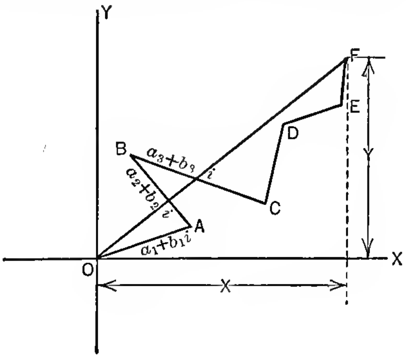


FIG. 103.

1. Test $1 + i + \frac{(1+i)^2}{1 \cdot 2} + \frac{(1+i)^3}{1 \cdot 2 \cdot 3} + \cdots$ for convergence.

Note

$$S = 1 + i + \frac{1 + 2i - 1}{1 \cdot 2} + \frac{1 + 3i - 3 - 1}{1 \cdot 2 \cdot 3} + \cdots,$$

$$X = 1 - \frac{3}{1 \cdot 2 \cdot 3} + \cdots,$$

$$Y = 1 + \frac{2}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} + \cdots.$$

2. For what values of x and y is the series

$$S = x + iy + \frac{(x+iy)^2}{2} + \left(\frac{x+iy}{3}\right)^3 + \cdots$$

convergent.

157. Expansion of functions in series of powers of the variable. — A series each of whose terms contains a power of the variable is called a power series. When the powers increase with n the series is called an ascending power series. If the powers decrease with n the series is a descending power series. A series containing only positive integral powers of the variable is called an integral power series.

To expand a function it is first assumed that the expansion is possible. If then the coefficients of the terms can be determined, the expansion is known to exist. But the resulting series must be tested for convergence before it can be used in calculations. There is but one power series that can represent a given function in the interval of convergence. By interval of convergence is meant the aggregate of values of the variable for which the series is convergent. If the interval is continuous over a finite region it is sufficiently designated by giving the bounding values.

158. Consider $F(x)$ any continuous function of x having derivatives of all orders which are continuous in a given interval.

Let

$$(1) \quad F(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots + Nx^n + \cdots$$

be assumed to hold for a certain interval of x. The values of the coefficients A, B, C, . . . are now to be determined. Calculate:

(2) $F'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots + Nnx^{n-1} + \cdots,$

(3) $F''(x) = 2C + 2 \cdot 3Dx + 3 \cdot 4Ex^2 + \cdots Nn(n-1)x^{n-2} + \cdots,$

(4) $F'''(x)^* = 2 \cdot 3D + 2 \cdot 3 \cdot 4Ex + \cdots$
$+ Nn(n-1)(n-2)x^{n-3} + \cdots,$

$\vdots \qquad \vdots$

From the form of the above equations it is evident they must hold for $x = 0$, since A, B, C, . . . are finite constants by hypothesis, provided $x = 0$ is in the region of convergence of (1). Therefore,

$$F(0) = A,$$
$$F'(0) = B,$$
$$F''(0) = 2C \quad \text{or} \quad C = \frac{F''(0)}{2},$$
$$F'''(0) = 3 \cdot 2D \quad \text{or} \quad D = \frac{F'''(0)}{2 \cdot 3}.$$
$$\vdots \qquad \vdots$$

Substituting these values of A, B, C, . . . in (1) gives

(5) $$F(x) = F(0) + F'(0)x + \frac{F''(0)}{2}x^2 + \frac{F'''(0)}{2 \cdot 3}x^3 + \cdots$$
$$+ \frac{F^n(0)}{2 \cdot 3 \cdots n}x^n + \cdots.$$

In this form $F(x)$ is said to be expanded about the origin.

If it is assumed that

(1′) $$F(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + \cdots + N(x-a)^n + \cdots,$$

* By $F'''(x)$ is meant the derivative of $F''(x)$. $F''(x)$ is the derivative of the second order of $F(x)$, $F'''(x)$ is the derivative of the third order, etc.

then

$$(2') \quad F'(x) = B + 2C(x-a) + 3D(x-a)^2 + \cdots.$$

$$(3') \quad F''(x) = 2C + 2 \cdot 3D(x-a) + \cdots.$$

$$(4') \quad F'''(x) = 2 \cdot 3D + \text{terms in } (x-a).$$

In these equations put $x = a$, and solve for $A, B, C, \ldots$ as above. Substitute the values of $A, B, C, \ldots$ in (1′) and obtain

$$(5') \quad F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2}(x-a)^2 + \frac{F'''(a)}{2 \cdot 3}(x-a)^3 + \cdots.$$

In this expansion, a must lie in the region of convergence of (1′). In (5′) $F(x)$ is said to be expanded about the point a.

If in (5′), $(x + a)$ be put for x there results

$$(6) \quad F(x+a) = F(a) + F'(a)x + \frac{F''(a)}{2}x^2 + \frac{F'''(a)}{2 \cdot 3}x^3 + \cdots.$$

In (6), exchange places with x and a and the result is

$$(7) \quad F(a+x) = F(x) + F'(x)a + \frac{F''(x)}{2}a^2 + \frac{F'''(x)}{2 \cdot 3}a^3 + \cdots.$$

Each expansion must be tested for convergence to be sure it can be used in calculations.

159. Consider the binomial series. — Assume

$$(1) \quad (a+x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots + Nx^n + \cdots,$$

where n is any number whatever. Now, by **158**,

$$A = a^n, \quad B = na^{n-1}, \quad C = \frac{n(n-1)}{1 \cdot 2},$$

$$N = \frac{n(n-1) \cdots (n-r+1)}{1 \cdot 2 \cdot \cdots \cdot r} a^{n-r}, \ldots.$$

Hence on substituting in (1)

$$(2) \quad (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot r}a^{n-r}x^r + \cdots.$$

If n is a finite positive integer the series (1) and (2) are finite. If n is not a positive integer the series are infinite series and (2) must be tested for convergence. We shall apply the ratio test. The $(r+1)$st term is

$$u_{r+1} = \frac{n(n-1)\ .\ .\ .\ (n-r+1)}{1\cdot 2\cdot 3\cdot\ \cdot\ \cdot\ \cdot\ \cdot r}a^{n-r}x^r;$$

the rth term is

$$u_r = \frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+2)}{1\cdot 2\cdot 3\cdot\ \cdot\ \cdot\ \cdot\ \cdot (r-1)}a^{n-r+1}x^{r-1};$$

the ratio

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}\frac{x}{a}.$$

The limit of this ratio, as $r \to \infty$, is $-\dfrac{x}{a}\cdot$ Hence the series is convergent if $\left|\dfrac{x}{a}\right| = k < 1$, that is, if $|x| < |a|$.

1. Find the value of $\sqrt{31}$. Write

$$\sqrt{31} = \sqrt{36-5} = (36-5)^{\frac{1}{2}} = 6\left(1-\frac{5}{36}\right)^{\frac{1}{2}}$$

$$= 6\left(1 - \frac{1}{2}\cdot\frac{5}{36} + \cdot\ \cdot\qquad\right) = \frac{6\,(72-5)}{72} = 5.59 \text{ (approx.)}.$$

Again

$\sqrt{31} = \sqrt{25+6} = (25+6)^{\frac{1}{2}} = 5\,(1+\frac{6}{25})^{\frac{1}{2}} = 5\,(1+\frac{1}{2}\cdot\frac{6}{25}) = 5.6$ (approx.).

Note. — More accurate results may be obtained by calculating more terms of the series. These expansions are convergent. For in each case the second term of the binomial is less than the first, that is, $\frac{5}{36} < 1$ and $\frac{6}{25} < 1$, which correspond to $|x| < |a|$ in the formula (2) above.

2. Find $\sqrt[3]{10} = \sqrt[3]{8+2}$.
4. Find $\sqrt{33}$.
3. Find $\sqrt[4]{18} = \sqrt[4]{16+2}$.
5. Find $\sqrt[5]{40}$.

160. Consider the expression:

$$(1)\quad F(x) = \left(1 + \frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{n(n-1)}{1 \cdot 2}\frac{x^2}{n^2} + \cdots$$
$$+ \frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (r-1)}\frac{x^{r-1}}{n^{r-1}} + \cdots.$$

Taking the limit of each term as $n \to \infty$ the series becomes

$$(2)\quad f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \cdots + \frac{x^r}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot r} + \cdots.$$

Equation (2) defines the function $f(x)$ for values of x for which the series converges. It is to be noted that the properties of the function are derivable from the series. By the ratio test

$$\frac{u_{r+1}}{u_r} = \frac{\dfrac{x^{r+1}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (r+1)}}{\dfrac{x^r}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot r}} = \frac{x}{r+1}.$$

The limit of this ratio as $r \to \infty$ is 0, **41**, I, if x is finite. Hence series (2) is convergent for all finite values of x and $f(x)$ is a continuous function for all finite values of x.

The function $f(x)$ will be called e^x (exponential of x) so that

$$(2')\quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \cdots + \frac{x^r}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot r} + \cdots.$$

When $x = 1$, there results

$$e = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \cdots = 2.718281 \ldots.$$

Eleven terms are necessary to obtain this value. The student should make this calculation in full.

161. Theorem on logarithms. — $\operatorname{Log}_b y = \log_b a \cdot \log_a y$, where a, b, y, satisfy the definitions of **19**, **20**.

Let	$\log_a y = u$ and $\log_b y = v$.
Then	$y = a^u$ and $y = b^v$.
Therefore	$a^u = b^v$.
	$a = b^{v/u}$.
	$\log_b a = v/u = \log_b y/\log_a y$.
That is,	$\log_b y = \log_a y \cdot \log_b a$.

This is exactly the theorem. This equation makes it possible to change the base of a system of logarithms. The two bases in common use are 10 and $e = 2.718281 \ldots$.

Example. — Let $b = e$, $a = 10$, $y = 25$ and $\log_{10} 25 = 1.3979$. To find $\log_e 25$.

By the above theorem write,

$$\log_{10} 25 = \log_e 25 \cdot \log_{10} e,$$

whence
$$\log_e 25 = \log_{10} 25 / \log_{10} e$$

or
$$\log_e 25 = 1.3979/0.4343 = 3.193.$$

1. Using the tables for finding logarithms to the base, 10, by above method calculate the logarithms of 20, 15, 35, 75, to the base e.

The number, $\log_e 10 = 0.4343$, which as a multiplier would change $\log_e y$ to $\log_{10} y$, is called the **modulus** of the system of logarithms to the base, 10.

162. The series (2′), **160**, may be used to obtain the derivatives of exponential function and the logarithmic function. Write

$$y = e^u, \tag{1}$$

where u is a function of some variable, say x. Then

$$y + \Delta y = e^{u+\Delta u} = e^u \cdot e^{\Delta u},$$
$$\Delta y = e^u (e^{\Delta u} - 1),$$

$$\frac{\Delta y}{\Delta x} = e^u \frac{\left(1 + \Delta u + \frac{(\Delta x)^2}{2} + \cdots + \frac{(\Delta x)^n}{1 \cdot 2 \cdot \cdots \cdot n} \cdots - 1\right)}{\Delta x}$$

$$= e^u \left(\frac{\Delta u}{\Delta x} + \Delta u \cdot \frac{\Delta u}{\Delta x} + \cdots + \frac{(\Delta u)^{n-1}}{1 \cdot 2 \cdot \cdots \cdot n} \frac{\Delta u}{\Delta x} + \cdots\right).$$

$$\frac{dy}{dx} = e^u \frac{du}{dx}, \tag{2}$$

since all terms after the first vanish with Δx. Now by (1),

$$\log_e y = u. \tag{3}$$

By rearranging (2) and using (1),

$$\frac{du}{dy} = \frac{1}{e^u} \frac{dx}{dx} = \frac{1}{y}. \tag{4}$$

Therefore

$$\frac{d}{dy}\log_e y = \frac{1}{y}. \tag{5}$$

If a is the base of the system of logarithms, multiply both sides of (4) by $\log_a e$, **161**, and there results

$$\frac{d}{dy}\log_a y = \log_a e \cdot \frac{1}{y}.$$

1. From a table of logarithms to the base e, construct the graph of the equation $y = \log_e x$.

2. From a table of logarithms to the base 10, construct the graph of the equation $y = \log_{10} x$. Compare the graphs of this exercise with the graph of Ex. 1. Discuss both.

3. Construct the graphs of $y = e^x$ and $y = \log x$ on the same axes to the same scale. Note the positions of the curves. Each is the image of the other reflected in a mirror placed at an angle of 45° with the x-axis. Such functions are inverse to each other. See **70**, **71**, graphs of $\sin x$ and $\text{arc}\sin x$.

4. Construct graphs of $y = x^2$ and $y = x^{\frac{1}{2}}$ on same axes and compare results with above.

163. It is often convenient to use the logarithmic function in calculating derivatives of algebraic functions. Consider:

$$y = \frac{x^m}{(1-x)^n}. \tag{1}$$

$$\log_e y = m\log_e x - n\log_e (1-x). \tag{2}$$

$$\frac{1}{y}\frac{dy}{dx} = m\frac{1}{x} + n\frac{1}{1-x}. \tag{3}$$

$$\frac{dy}{dx} = \frac{y\,[m-(m-n)\,x]}{x\,(1-x)}. \tag{4}$$

$$= -\frac{(m-n)\,x^m - mx^{m-1}}{(1-x)^{n+1}}.$$

1. Find the derivative of $y = x/(1-x)$, by above method.
2. Find the derivative of $y = (1-a^x)^2$, by above method.
3. Find the derivative of $y = e^x x^n$, by above method.

4. Find the derivative of $y = u^v$, where u and v are functions of x. Save result as a formula.

5. Find the derivative of $y = \frac{1 + x^2}{1 - x^2}$.

6. Calculate the tabular difference for logarithms of numbers between 120 and 121. (Use Eq. 5, **162.**)

164. It is now quite easy to derive a series by means of which the logarithms of numbers may be calculated. Using Eq. (7), **157**, write

$$(1)\quad f(x + a) = \log_e (x + a) = f(x) + f'(x) \cdot a + \frac{f''(x)}{2} a^2 + \frac{f'''(x)}{2 \cdot 3} a^3 + \cdots .$$

The derivatives are

$$f(x) = \log_e x,$$

$$f'(x) = \frac{1}{x},$$

$$f''(x) = -\frac{1}{x^2},$$

$$f'''(x) = \frac{2}{x^3},$$

$$\vdots \qquad \vdots$$

Substituting in (1) gives:

$$(2)\qquad \log_e (x + a) = \log_e x + \frac{a}{x} - \frac{a^2}{2x^2} + \frac{a^3}{3x^2} - \cdots .$$

If $x = 1$ this becomes,

$$(3)\qquad \log_e (1 + a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots .$$

$$(4)\qquad \text{Write also } \log_e (1 - a) = -a - \frac{a^2}{2} - \frac{a^3}{3} - \frac{a^4}{4} - \cdots .$$

$$(5)\qquad \therefore\quad \log_e \left(\frac{1 + a}{1 - a}\right) = 2\left(a + \frac{a^3}{3} + \frac{a^5}{5} + \cdots\right).$$

Now put $a = \frac{1}{2m + 1}$ in Eq. (5) and transpose

$$(6)\qquad \log_e (m+1) = \log_e m + 2\left(\frac{1}{2m + 1} + \frac{1}{3(2m + 1)^3} + \cdots\right).$$

This series converges rapidly and three or four terms are sufficient for calculating logarithms of ordinary numbers.

1. Test series (6) for convergence.

2. In (6) put $m = 1, 2, 3$, in succession, using three terms, multiply each result by $\log_{10} e = 0.4343$ and compare the results with the logarithms of 2, 3, in the tables.

3. Given $\log_{10} 10 = 1$, calculate by use of above series log 11.

4. Given $\log_{10} 3$, $\log_{10} 4$, taken from the tables, calculate $\log_{10} 13$.

165. Consider $$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{2\cdot 3}{}^{*} + \cdots$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4} - \cdots$$
$$+ i\left(x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \cdots\right),$$
where $i^2 = -1$. Now write

$$f(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{1}$$

and

$$f_1(x) = \frac{e^{ix} + e^{-ix}}{2}. \tag{2}$$

Squaring these by actual multiplication and adding gives

$$[f(x)]^2 + [f_1(x)]^2 = 1.$$

Compare this result with equation (1), **48.** Again carry out the work and obtain from (1), (2) above,

$$f(x)\cdot f_1(y) + f_1(x) f(y) = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} = f(x+y).$$

Compare this result with equation 24, **53.** These two relations are sufficient to show that $f(x)$ and $f_1(x)$ are the sine and cosine functions. But which is sine and which is cosine is not yet determined. Substituting $-x$ for x in $f(x)$, the result is

$$f(-x) = -f(x).$$

* Here as in a previous theorem we have assumed i to be a symbol treated as a number.

Therefore $f(x)$ is an odd function like the sine function. By the same substitution,

$$f_1(-x) = f_1(x)$$

and $f_1(x)$ is an even function like the cosine function. It is safe to conclude that

$$f(x) = \sin x,$$
$$f_1(x) = \cos x.$$

The right members of (1) and (2) are known as Euler's expressions for the sine and cosine functions, respectively.

166. The derivatives of the sine and cosine functions can be obtained by use of (1), (2), **165.** Write

$$y = \sin u = \frac{e^{iu} - e^{-iu}}{2i}.$$

Then
$$\frac{dy}{dx} = \frac{d}{dx}\sin u = \frac{1}{2i}\left(ie^{iu}\frac{du}{dx} + ie^{-iu}\frac{du}{dx}\right)$$
$$= \frac{(e^{iu} + e^{-iu})}{2}\frac{du}{dx}$$
$$= \cos u \frac{du}{dx}.$$

(1) $$\therefore \quad \frac{d}{dx}\sin u = \cos u \frac{du}{dx}.$$

Similarly
$$y = \cos u = \frac{e^{iu} + e^{-iu}}{2}.$$

$$\frac{dy}{dx} = \frac{d}{dx}\cos u = \frac{ie^{iu}\frac{du}{dx} - ie^{-iu}\frac{du}{dx}}{2}$$
$$= -\frac{(e^{iu} - e^{-iu})}{2i}\frac{du}{dx}.$$

(2) $$\therefore \quad \frac{d}{dx}\cos u = -\sin u \frac{du}{dx}.$$

By use of $\tan u = \dfrac{\sin u}{\cos u}$ derive

(3) $$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}.$$

1. Derive $\dfrac{d}{dx}\cot u = -\csc^2 u\,\dfrac{du}{dx}\cdot$

2. Using $\sec u = \dfrac{1}{\cos u}$, derive $\dfrac{d}{dx}\sec u = \sec u \tan u\,\dfrac{du}{dx}\cdot$

3. Using $\csc u = \dfrac{1}{\sin u}$, derive $\dfrac{d}{dx}\csc u = -\csc u \cot u\,\dfrac{du}{dx}\cdot$

4. From $y = \sin^2 u$, find dy/dx.

5. From $y = \sin 2u = 2\sin u \cos u$, find dy/dx and show that $\cos 2u = \cos^2 u - \sin^2 u$.

6. From $y = 2\sin 2u + 3\cos u$, find dy/dx.

7. From $y = \tan u + u$, find dy/dx.

8. From $y = \log \sin u$, find dy/dx.

9. From $y = \log \sec u$, find dy/dx.

10. From $y = (a \sec^2 u + b \cos^2 u)^3$, find dy/dx.

11. From $y = e^{\sin x}$, find dy/dx.

12. From $y = \tan^2 u + \dfrac{1}{u^2}$, find dy/dx.

13. From $y = \sin u \cos v + \cos u \sin v$, find dy/dx. Regard each term as a product of two functions.

14. From $y = \dfrac{3 \sin u}{\cos 2u}$, find dy/dx.

15. From $y = \sin 3u + 6\cos 2u$, find dy/dx.

16. From $y = 2\sin\dfrac{x}{2} - 3\tan\dfrac{x}{2}$, find dy/dx.

167. To expand $\sin u$ and $\cos u$ in power series, write by **157** (5)

$$(1)\quad f(u) = \sin u = \sin 0 + \left.\frac{d}{du}\sin u\right]_{u=0} u + \frac{\left.\dfrac{d^2}{du^2}\sin u\right]_{u=0}}{2}u^2 + \cdots.$$

Now
$$\frac{d}{du}\sin u = \cos u,$$
$$\frac{d^2}{du^2}\sin u = \frac{d}{du}\cos u = -\sin u,$$
$$\frac{d^3}{du^3}\sin u = \frac{d}{du}(-\sin u) = -\cos u,$$
$$\frac{d^4}{du^4}\sin u = \frac{d}{du}(-\cos u) = \sin u.$$

Setting $u = 0$ in each of these and substituting the values in (1) gives

$$\sin u = u - \frac{u^3}{2 \cdot 3} + \frac{u^5}{2 \cdot 3 \cdot 4 \cdot 5} - \cdots .$$

1. Test the last series for convergence. Compare the last series with the real part of the expansion of e^{ix}, **161.**

2. In a manner similar to the above expand $\cos u$ in a power series. Compare the result with the imaginary part of the expansion for e^{ix}, **161.** Test the last series for convergence.

Note in using derivatives and series of the trigonometric functions the variable must be expressed in radians.

3. Using the series obtained for $\sin u$, put $u = \frac{\pi}{6}$ $(\pi = 3.1416)$ and calculate the $\sin \frac{\pi}{6} = \sin 30°$. Compare the result with the value in a table of natural sines.

4. Using the results obtained above can you now prove that

$$e^{ix} = \cos x + i \sin x?$$

5. Plot graph of and discuss $y = e^{x^2}$, find slope at $x = -1$, $x = 1$.

6. Plot graph of and discuss $y = e^{-x^2}$, find slope at $x = -1$, $x = 1$.

7. Plot graph of and discuss $\begin{cases} x = r\theta - r \sin \theta, \\ y = r - r \cos \theta, \ r \text{ constant}, \end{cases}$ x, y, the coördinates of points on the curve for values of θ. This curve is the cycloid. Read up in an encyclopedia on this very remarkable curve.

8. Find the slope of the cycloid at $\theta = 0$, $\theta = \pi/4$, $\theta = \pi$.

Note. — Use $$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}.$$

9. Construct and discuss $y = \sin x + \cos x$.

10. Construct and discuss $y = e^{-x} \sin x$; find slope at $x = \pi/2$.

11. Find the angle at which $y = \sin x$ and $y = \cos x$, intersect.

12. Find the equation of the tangent to $y = \sin x$, at $x = \pi/6$, $x = \pi/2$.

13. Construct and discuss $r = ae^{n\theta}$ (polar coördinates).

14. Construct and discuss $r = \cos 3\,\theta$ (polar coördinates).

15. Construct and discuss $y = \cos 3\,x$ (rectangular coördinates).

167a. In certain classes of problems where the formulas to be determined are of the form $y = Ax^n$, it is often easier to determine the constants from a graph constructed with the logarithms of the number pairs instead of the numbers themselves. This can be done by use of a table of logarithms and ordinary coördinate paper. Of such frequent use is this method in certain branches of engineering that special coördinate paper is used. This paper is divided and ruled in spaces proportional to the logarithms of numbers as is the slide rule. It is then only necessary to plot with this paper as with ordinary paper. Such paper is called logarithmic coördinate paper.

Suppose we have the number pairs,

	$x =$	1	1.5	2	3	10
	$y =$	4	9	16	36	400
then	$\log x =$	0	0.176	0.301	0.477	1
and	$\log y =$	0.602	0.954	1.204	1.556	2.602

Locating the x, y number pairs directly on the logarithmic diagram, we obtain the line AB. To determine n measure DC to any scale and $4\,C$ to the same scale. The ratio $\frac{DC}{4\,C} = 2$ is the slope of AB and is the value of n. The value of A is the number corresponding to $\overline{1\,4}$ or 4. Now substituting these values in the proposed equation $y = Ax^n$ there results

$$y = 4\,x^2$$

as the desired equation.

To see that these values are thus determined write the proposed equation in the form

$$\log y = \log A + n \log x.$$

As log y and log x are the coördinates of points in this method log A is the y-intercept of the line and n is the slope.

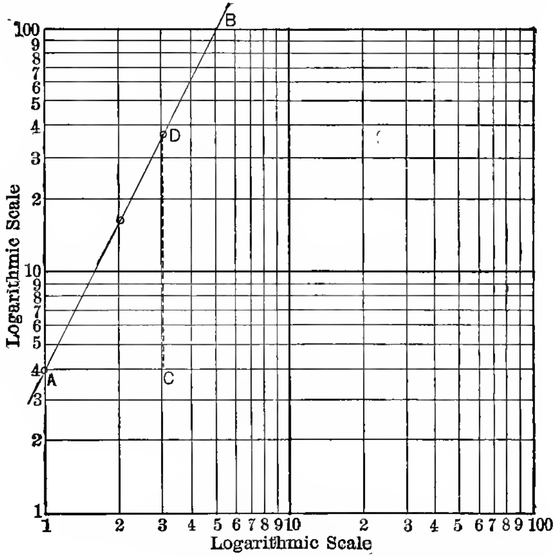


FIG. 104.

The student should procure several sheets of logarithmic coördinate paper and solve the above problem completely. It should be noted that the divisions on this paper are exactly those on the A scale of the Mannheim slide rule.

1. Draw the graph of the above problem on ordinary coördinate paper and on logarithmic paper.

Interpolate several points on each graph and determine their coördinates from each graph. Which of the graphs affords the easiest and most accurate interpolation? It will be seen that the logarithmic graph has decided advantages.

2. Make the graph on logarithmic paper and determine an equation of the form $y = Ax^n$ from the data below and also construct the ordinary graph.

$x = 1$	2	3	4	5
$y = 2$	8	18	32	50

3. Construct the logarithmic graph and determine the equation as directed in (2) from the data below.

$x = 10$	30	40	60	80	100
$y = 86$	61.5	45	34	26	21.5

Remark. — It should be noted that the logarithmic coördinate paper provides directly only for numbers from 1 to 100. The scale is often 5 in. to 1 unit. Each of the two main divisions of the paper provides for numbers whose logarithms have the same characteristic. If numbers from 100 to 1000 are to be represented, the axes, main division lines of the paper, must be renamed. Ordinarily the left margin is named 1, the middle line 10 and the right margin 100. The axis 1 may be called 10 or 100, and the others correspondingly. This is equivalent to changing the characteristic without disturbing the mantissa of the logarithms. To illustrate this the lower margin of the paper with axes renamed is shown in the figure below.

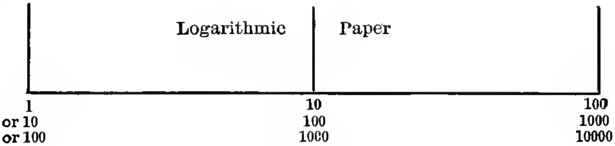


FIG. 105.

167b. Illustrative problem 1. — The electrical resistance of a certain river water when different amounts of solids were in solution was found to be as follows:

Resistance (ohms)	$y =$	1000	800	600	400	300	200.
Solids (parts in 10,000,000)	$x =$	215	260	340	480	615	860.

On the logarithmic paper let K be the point (1000, 100) at first. Lay off the points from the data of the problem; they determine the line BAA^{I}. Drop the point A^{I} to A^{II} and then move A^{II} to A^{III}. Draw $A^{\mathrm{III}}A^{\mathrm{IV}}$ parallel to AA^{I}. The new names of the axes appear underscored on the diagram. The point A^{IV} is the y-intercept of the line. For $A^{\mathrm{III}}A^{\mathrm{IV}}$ is in reality a continuation of BAA^{I}, and A^{IV} read to the new-named axes is

the point where the line crosses the axis originally named 1. Since the graph is a straight line on the logarithmic paper the equation desired will be of the form $y = Ax^n$. Proceeding as in the illustrative problem of **34**, the values of A and n are

$$A = 300{,}000 \text{ and } n = -1.05.$$

The equation is, therefore,

$$y = 300{,}000\, x^{-1.05}.$$

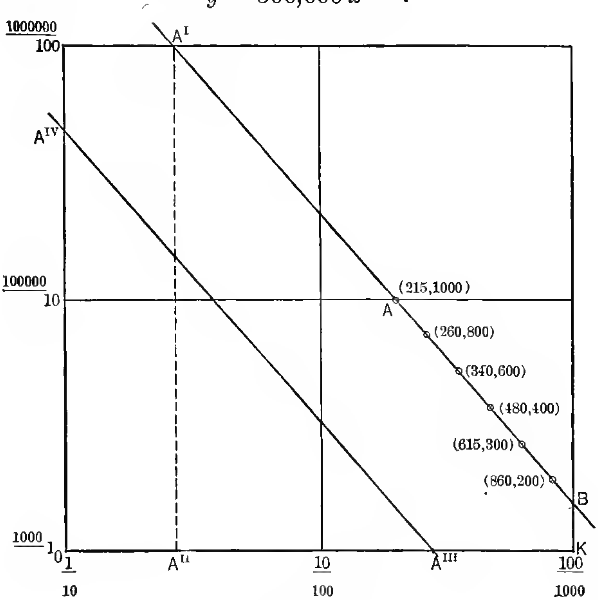


FIG. 106.

Note. — The sliding of the graph downward and to the left is done to avoid having to extend the drawing off the paper to the left and upward. The slope of BA^{I} is negative because the line slopes to the left and upward. This will be further explained later.

167c. Illustrative problem 2. — Let us determine the equation of problem 4, **35**, by a method similar to that of the last section. Taking logarithms of both sides of the equation gives

$$\log p = \log A + (K \log e)\, h.$$

This equation is of the first degree in $\log p$ and h. This suggests making the graph with a logarithmic scale for p and an ordinary scale for h. It is evident the slope of the line is $K \log e$ and the intercept on the p-axis is $\log A$. The point A is 30. Measure AD to the scale 2″ to 1 unit and DC to the scale 2000 to 1″. Then the slope of CA is $\dfrac{-DA}{CD} = \dfrac{-0.5}{8000} = 0.0000625$. The slope is also $K \log e$ and $\log e = 0.4343$.

Hence
$$K = -\frac{0.0000625}{0.4343} = -0.00144.$$

Hence the equation is
$$p = 30\, e^{-0.00144\, h}.$$

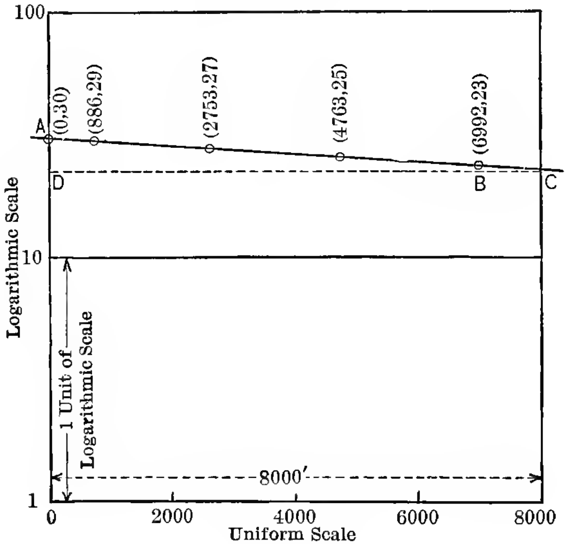


FIG. 107.

The paper ruled to logarithmic scale one way and to ordinary scale the other way, as used in this problem, is called semi-logarithmic paper.

1. Construct on semilogarithmic paper the graph and determine an equation of the form

$$y = Ae^{kx}$$

from the data below. A thermometer initially at 19.3° C. is exposed to the air and the readings taken afterward as follows:

Time (seconds after starting) $t =$	0	20	40	60	(uniform scale)
Temperature C =	19.3	14.2	10.4	7.6	(log. scale)

2. Construct the logarithmic graph and determine the equation from the following: The amount of water that will flow through a certain length of pipe of different diameters under a pressure of 50 lbs. per square inch is given below:

Diameter in ft. $d =$	1	1.5	2	3	4	6
Quantity water cu. ft. per sec. $q =$	4.88	13.43	27.5	75.13	152	409

3. Given the number pairs:

$x =$	0	1	2	3	4
$y =$	0	$\frac{1}{10}$	$\frac{8}{10}$	2.7	6.4

Draw graph on logarithmic paper and determine the equation.

4. The following data were observed on a given amount of gas when no heat was allowed to enter or escape:

Pressure $p =$	20.5	25.8	54.2
Volume $v =$	6.3	5.3	3.2

Plot on logarithmic paper and determine the relation between p, v in the form of an equation.

CHAPTER XVII

INTEGRATION

168. The derivatives of several types of functions have been considered. Some of the applications of derivatives have been illustrated. It will be profitable now to consider the inverse problem, *viz.*: Given the derivative of a function, to determine the function. This problem and the process involved are included under the name **integration.** The sign of integration is "$\int$." When this sign stands before an expression it indicates that a new function is to be determined whose derivative is the expression under (immediately following) the sign of integration. Integration enables us to solve a great number of new problems in science and geometry.

Consider, $\int 3\,x^2 = x^3 + C$, where C is any constant. To prove this equation, take the derivative of the right side and compare with the expression under the sign of integration. This derivative is seen to be exactly $3x^2$, which proves the truth of the equation. C is called the constant of integration and cannot be determined without further information. The expression $x^3 + C$ is called the integral of the expression under the sign of integration, $3x^2$. In what follows, it is assumed that the integrand* is continuous for values of the variable considered.

For convenience of reference the derivatives of a few fundamental functions and the corresponding integrals are given:

* The integrand is the expression under the sign of integration.

Derivative.	Inverse derivative.	Integral of differential.
1. $\frac{d}{dx} C = 0$	$\int 0 = C$	$\int 0\, dx = C$
2. $\frac{d}{dx} Cu = C \frac{du}{dx}$	$\int C \frac{du}{dx} = Cu + C_1$	$\int C \frac{du}{dx} dx = Cu + C_1$
3. $\frac{d}{dx} x = 1$	$\int 1 = x + C$	$\int dx = x + C$
4. $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$	$\int u^n \frac{du}{dx} = \frac{u^{n+1}}{n+1} + C$	$\int u^n \frac{du}{dx} \cdot dx = \frac{u^{n+1}}{n+1} + C$
5. $\frac{d}{dx} uv = v \frac{du}{dx} + u \frac{dv}{dx}$	$\int v \frac{du}{dx} + u \frac{dv}{dx} = uv + C$	$\int \left(v \frac{du}{dx} + u \frac{dv}{dx}\right) dx = uv + C$
6. $\frac{d}{dx} \left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$	$\int \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{u}{v} + C$	$\int \left(\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}\right) dx = \frac{u}{v} + C$
7. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$	$\int e^u \frac{du}{dx} = e^u + C$	$\int e^u \frac{du}{dx} \cdot dx = e^u + C$
8. $\frac{d}{dx} a^u = a^u \log a \frac{du}{dx}$	$\int a^u \frac{du}{dx} = \frac{a^u}{\log a} + C$	$\int a^u \frac{du}{dx} \cdot dx = \frac{a^u}{\log a} + C$
9. $\frac{d}{dx} \log_e u = \frac{1}{u} \cdot \frac{du}{dx}$	$\int \frac{1}{u} \cdot \frac{du}{dx} = \log_e u + C$	$\int \frac{1}{u} \cdot \frac{du}{dx} \cdot dx = \log_e u + C$
10. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$	$\int \cos u \frac{du}{dx} = \sin u + C$	$\int \cos u \frac{du}{dx} \cdot dx = \sin u + C$
11. $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$	$\int \sin u \frac{du}{dx} = -\cos u + C$	$\int \sin u \frac{du}{dx} \cdot dx = -\cos u + C$
12. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$	$\int \sec^2 u \frac{du}{dx} = \tan u + C$	$\int \sec u \frac{du}{dx} \cdot dx = \tan u + C$
13. $\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$	$\int \csc^2 u \frac{du}{dx} = -\cot u + C$	$\int \csc^2 u \frac{du}{dx} \cdot dx = -\cot u + C$
14. $\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$	$\int \left(\frac{du}{dx} + \frac{dv}{dx}\right) = \int \frac{du}{dx} + \int \frac{dv}{dx} + C$	$\int \left(\frac{du}{dx} + \frac{dv}{dx}\right) dx = \int \frac{du}{dx} dx + \int \frac{dv}{dx} \cdot dx + C$

The chief difficulty in integration is to recognize the type form of the integrand and the factor to be used as $\frac{du}{dx}$. To factor and arrange an integrand to fit an integration formula, it is often necessary to introduce or take out a constant factor. The selection of the formula of integration is a matter of judgment and experience. To illustrate, take

$$\int xe^{3x^2} = \tfrac{1}{6}\int e^{3x^2}\cdot 6x = \tfrac{1}{6}e^{3x^2} + C \text{ (formula 7).}$$

To verify the result find the derivative of the integral. This is precisely the original integrand xe^{3x^2}.

A variable factor must not be introduced under the integral sign nor taken from under the integral sign.

Consider $\int (a + bx^2)^3 x$. This may be written:

$$\int (a+bx^2)^3 x = \frac{1}{2b}\int (a+bx^2)^3\, 2bx = \frac{1}{2b}\,\frac{(a+bx^2)^4}{4} + C$$
$$= \frac{1}{8b}(a+bx^2)^4 + C.$$

1. $\int x^3$ * $= ?$
2. $\int x^{\frac{1}{2}} = ?$
3. $\int \frac{1}{x} = ?$
4. $\int (1+3x)^2 = ?$
5. $\int \frac{b}{a+bx} = ?$
6. $\int (x^2+4)^2 x = ?$
7. $\int (x^2+1)(1+x) = ?$
   (multiply)
8. $\int \frac{x^3+3x^2+8x-4}{x^2} = ?$
9. $\int e^{2x} = ?$
10. $\int (e^{2x} + e^{x/2}) = ?$
11. $\int \frac{1}{x \log x} = ?$
12. $\int \frac{x}{1+x^2} = ?$
13. $\int \frac{a(x-b)^3}{px^2} = ?$
14. $\int 3a^x = ?$

* Each integrand may be multiplied by dx before integrating, if desired. See **175.**

15. $\int \frac{\sin 3x}{\cos^2 3x} = \int \cos^{-2} 3x \sin 3x = ?$

16. $\int \sin^3 x \cos x = ?$

17. $\int (1 + 2x)(x + x^2) = ?$

18. $\int (1 - \cos 2x) = ?$

19. $\int \tan x \sec^2 x = ?$

20. $\int \cot^2 x \csc^2 x = ?$

21. $\int \csc^2 x = ?$

22. $\int (1 + \tan^2)^2 (\sec^2 x) = ?$

23. $\int \frac{\sec^2 x}{\tan x} = ?$

24. $\int \csc^3 x \cot x = ?$

25. $\int x \cos x^2 = ?$

26. $\int \cos x \cdot e^{\sin x} = ?$

27. $\int (1 - x^2)^{\frac{1}{2}} x = ?$

28. $\int \frac{x}{1 + 3x^2} = ?$

29. $\int \left(1 + \frac{1}{x} + \frac{3}{x^2} + \frac{4}{x^2} + \sec^2 x\right) = ?$

30. $\int \frac{1}{(1 - x)^2} = ?$

31. $\int (1 + x)(1 - x) = ?$

32. $\int (x^3 - x^2 - x - 1) = ?$

33. $\int (1 - \sin x) \cos x = ?$

34. $\int \frac{x^2 - 1}{x + 1} = ?$

35. $\int e^{1-x} = ?$

36. $\int \cos (3x + 2) = ?$

37. $\int 6 \sin (4x^2 - 1) x = ?$

38. $\int (1 + \tan^2 x)^2 \sec^2 x = ?$

169. After some practice in the process of integration some easy applications can be taken up. The solution of a problem will consist of the following steps:

1. From the given data, formulate the correct integrand.
2. Select the formula for integrating.
3. Rearrange the integrand, introducing constant factors, if necessary, to make it fit the formula.
4. Integrate the expression.

170. The function obtained by integrating will have a distinct value for each value of the variable substituted in it. The difference between two values of the integral corresponding to two values of the variable is called a **definite integral.** Thus if

$$(1) \qquad \int x^4 = \frac{x^5}{5} + C$$

be evaluated for $x = 1$ and $x = 5$ and the former value subtracted from the latter there will result

$$(2) \qquad (5)^4 + C - \left[\frac{1^4}{5} + C\right] = 624\tfrac{4}{5}.$$

The values $x = 1$ and $x = 5$ are the limits of integration, the value $x = 5$ being the upper limit and the value $x = 1$ the lower limit. As a brief way to indicate what was done in (1) and (2) above we write

$$\int_1^5 x^4 = \frac{x^5}{5} + C\Bigg]_{x=1}^{x=5} = 624\tfrac{4}{5}.$$

Now consider the second illustrative problem of **170.** If $x = 0$ and $x = 4$ be taken as the limits of integration, there will result

$$\frac{1}{8\,b}(a + bx^2)^4 + C\Bigg]_{x=0}^{x=4} = \frac{1}{8\,b}(a + 16\,b)^4 + C - \left(\frac{1}{8\,b}\cdot a^4 + C\right)$$
$$= \frac{1}{8\,b}(a + 16\,b)^4 - a^4.$$

In the process of subtraction in each of the above cases the constant of integration C was eliminated. This will always occur. When the limits of integration are given it is unnecessary to determine C for it can be eliminated. An integral in which the undetermined constant of integration appears is called an indefinite integral. All the integrals of **168** are indefinite integrals.

1. Given the instantaneous speed of a point moving in a straight line as a function of the time, t, from some position of reference, $v = 1 + 6\,t^2$. Find the displacement function and the space passed over during the interval from the beginning of the 5th to the end of the 12th second.

We have $$v = \frac{ds}{dt} = 1 + 6\,t^2, \textbf{ 144.}$$

Hence $$s = \int (1 + 6\,t^2) = t + 2\,t^3 + C.$$

This is the displacement function with C undetermined. The space passed over is the value of the definite integral,

$$\int_4^{12} (1 + 6\,t^2) = t + 2\,t^3 + C\Big]_{t=4}^{t=12} = 3336.$$

2. In the above problem, instead of the given data, suppose that $t = 0$, when $s = 0$, and let us solve the problem with these conditions.

We have from above,

$$s = t + 2\,t^3 + C.$$

For $t = 0,\quad s = 0,\quad 0 = 0 + 0 + C$

and $C = 0.$

Hence $s = t + 2\,t^3$

is the complete space function. The second part of the problem is solved now as

$$s = t + 2\,t^3\Big]_{t=4}^{t=12} = 3336,$$

as above.

3. The acceleration of a particle is $a = kt^2$. Find the speed and displacement functions and determine the speed and displacement from rest to $t = t_1$. See **144.**

4. The speed of emptying a vessel by a hole at the bottom varies with the square root of the depth of the hole below the surface of the liquid. How long will it take to empty a cylinder 3′ in diameter, 10′ high, if the opening is 1″ in diameter? Given the speed of flow $v = \sqrt{2\,gh}$, where h' is the depth and $g = 32.16'$.

5. The speed of a body from rest is $v = 3\,t^2 - 6\,t + 10$. Find the acceleration at $t = 0$ and at $t = 5$. Find the distance passed over from rest until $t = 5$, $t = 10$.

171. Area under a curve. — The area under a curve is that portion of the coördinate plane bounded by the curve, the x-axis and the ordinates at the ends of that part of the curve considered. Under this definition portions of the area that may lie below the x-axis are regarded as negative. In the figure PQX_2X_1 is the area under the curve $PABQ$. An element or increment of area is defined as a narrow strip bounded by the arc θ, the ordinates at its ends and an element Δx of the x-axis.

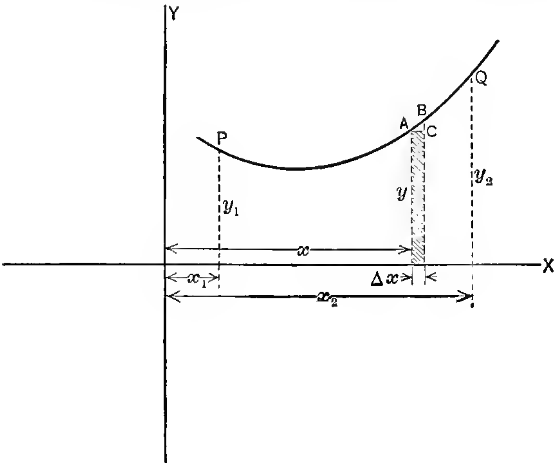


FIG. 108.

Now take $$\Delta A = \text{shaded area} + ACB$$
$$= y\,\Delta x + \theta\,\Delta y \Delta x,$$
where $$CB = \Delta y \quad \text{and} \quad \theta \lesseqqgtr 1.$$
If $$y = f(x)$$
is the equation of the curve, we have
$$\Delta A = f(x)\,\Delta x + \theta\,\Delta f(x) \cdot \Delta x,$$
$$\frac{\Delta A}{\Delta x} = f(x) + \theta\,\Delta f(x),$$
$$\frac{dA}{dx} = f(x) = y,$$
since $$\Delta f(x) = \Delta y \rightarrow 0 \quad \text{as} \quad \Delta x \rightarrow 0.$$

It follows that

$$A = \int_{x_1}^{x_2} y = \int_{x_1}^{x_2} f(x)$$

is the area under the curve.

1. Find the area under $y^2 = 12\,x$, between the ordinates for $x = 0$ and $x = 10$.

Write

$$A = \int_0^{10} y = \int_0^{10} \sqrt{12} \cdot x^{\frac{1}{2}} = \frac{\sqrt{12}\,x^{\frac{3}{2}}}{3/2}\Bigg]_{x=0}^{x=10} = 72.5.$$

2. Find the area under $y = x^2 + 2\,x - 1$, between $x = 0$, $x = 6$.

3. Find the area under $\frac{x}{3} + \frac{y}{4} = 1$ and the axes.

4. Find the area under $xy = 12$ from $x = 1$ and $x = 12$.

5. Find the area between $xy = 12$ and $x + y = 12$.

Note. — The area between the curves is the difference of the areas under the curves between the abscissas of the points of intersection.

6. Find the area under $y = \sin x$, from $x = 0$ to $x = \pi$. From $x = \pi$ to $x = 2\,\pi$.

7. Find the area of the triangle whose vertex is at 0, altitude, 16, lying on the x-axis and base 24, perpendicular to the x-axis.

8. Find area under $y = \frac{4}{x}$ from $x = 1$ to $x = 10$.

9. Find area under $y = e^x$ from $x = -\infty$ to $x = 0$.

10. Find by integration the area of a rectangle of base, b, and altitude, h.

172. Volumes of solids of revolution. — Let $y = f(x)$ be a curve. Let any arc of the curve, between two ordinates, revolve about the x-axis. The solid described is a solid of revolution. If an element of the curve, $AB = \Delta s$, revolve about the axis a thin lamina or slice is generated whose volume is

$$\begin{aligned} \Delta V &= \pi y^2\,\Delta x + 2\,\pi\theta\,\Delta x\,\Delta y \\ &= \pi\,[f(x)]^2\,\Delta x + 2\pi\theta\,\Delta x\,\Delta f(x), \end{aligned}$$

where $\theta \leqq 1$, and $\theta \rightarrow 0$ as $\Delta x \rightarrow 0$.

Hence

$$\frac{\Delta V}{\Delta x} = \pi\,[f(x)]^2 + 2\,\pi\theta\,\Delta f(x)$$

and

$$\frac{dV}{dx} = \pi\,[f(x)]^2.$$

Hence

$$V = \pi \int_{x_1}^{x_2} [f(x)]^2.$$

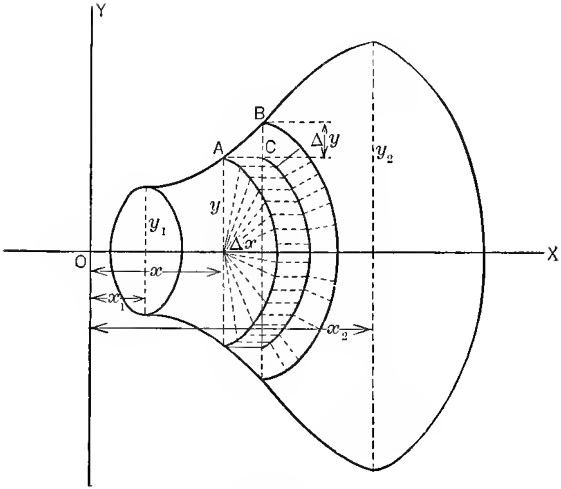


Fig. 109.

1. Find the volume generated by revolving an arc of $y = 6\,x + 1$, between $x = 0$, $x = 10$, about the x-axis. Write

$$\frac{dV}{dx} = \pi(6\,x + 1)^2.$$

$$V = \int_0^{10} \pi(6\,x + 1)^2$$

$$= \frac{\pi(6\,x+1)^3}{18}\Bigg]_{x=0}^{x=10}$$

$$= 12610\,\pi.$$

2. Find the volume generated by revolving $y = e^x$ about the x-axis between the limits $x = 0$, $x = 10$. Also between $x = -\infty$, $x = 0$.

3. Find the volume generated by revolving $y = \frac{1}{x}$ about the x-axis from $x = 1$ to $x = 10$.

4. Find the volume generated by revolving $y = \sin x$ about the x-axis from $x = 0$ to $x = \pi$.

Note. — Remember $2 \sin^2 x = 1 - \cos 2x$.

5. Find the volume generated by revolving $y = e^{x/2} - e^{-x/2}$ about the x-axis from $x = 0$ to $x = 4$.

6. Find the volume bounded by the two surfaces generated by revolving $xy = 12$ and $x + y = 12$ about the x-axis.

7. Find by integration the volume of a cylinder of radius, a, and altitude, h.

Note. — Consider the cylinder generated by revolving a rectangle.

8. Find the volume generated by revolving $y = 8x$ about the x-axis, from $x = 0$ to $x = 10$. From this result derive the rule for finding the volume of a cone when the radius and altitude are given.

9. Find the volume generated by revolving $y = x^3$ about the x-axis, from $x = 0$ to $x = 4$.

10. Find the volume generated by revolving $x^2 + y^2 = r^2$ about the x-axis, from $x = 0$ to $x = r$.

Note. — Solve the equation for y, $y = \sqrt{r^2 - x^2}$.

From the result derive the rule for finding the volume of a sphere of radius, r.

11. Find the volume generated by revolving $9x^2 + 16y^2 = 144$ about the x-axis.

12. A complete meridian of the earth is an ellipse whose long diameter is 7926 mi., and short diameter 7899 mi. Write, in standard form, the equation of this ellipse and by the method of Example 11, find the volume of the earth in cu. mi.

173. The **average value** of a function over an interval of the variable. Let $y = f(x)$ be the function and $x_1 < x < x_2$ the interval. From the figure it is evident the average value of the function is the average of the ordinates of the curve over the interval, and is the altitude of a rectangle whose base is

x_2x_1 and whose area is the area under the curve. Therefore, write

$$\frac{A}{x_2 - x_1} = y_a = \frac{\int_{x_1}^{x_2} f(x)}{x_2 - x_1} \text{ or } \left(\frac{A}{x_2 - x_1} = y_a = \frac{\int_{x_1}^{x_2} f(x)\,dx.}{x_2 - x_1}\right).$$

This expression gives the average value sought. This is an extension of the idea of average to an infinite number of values.

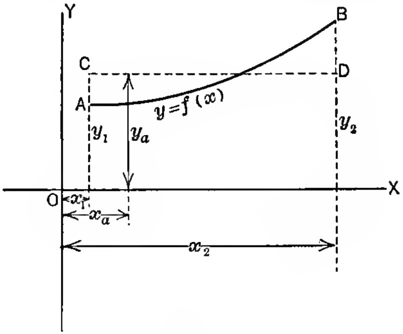


FIG. 110.

1. Find the average ordinate of $y = \sin x$ between $x = 0$ and $x = \pi$. Write

$$y_a = \frac{A}{\pi} = \frac{\int_0^{\pi} \sin x}{\pi}$$

$$= \frac{-\cos x}{\pi}\Bigg]_{x=0}^{x=\pi} = \frac{2}{\pi} = 0.63+.$$

Note. — This example has important bearing on the theory of dynamos and motors.

2. Find the average ordinate of $y = \cos x$ between $x = 0$ and $x = \pi$.

Note. — Remember areas below the x-axis are negative.

3. Find the average ordinate of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.

4. Find the altitude of a rectangle of base 12 that equals the area under $y = -x^2 + 6x + 27$. Draw figure.

5. Find the average ordinate of $y = a^2 - x^2$ between $x = -a$, $x = a$.

6. Find the average ordinate of $x + y = 12$ between $x = 0$ and $x = 12$.

7. Find the average ordinate of $y = \dfrac{1}{1+x}$ between $x = 0$ and $x = \frac{1}{2}$. Between $x = -2$ and $x = -1$. Between $x = -1$ and $x = 0$. Between $x = -2$ and $x = 0$. How do you explain these results? Draw figure.

8. Find the average ordinate of $y = e^{-x}$ between $x = -3$ and $x = 3$.

9. Find the average ordinate of $y = \cos x + \sin x$ between $x = \pi/2$ to $x = \pi$.

10. Find the average values of $y = x$, $y = x^2$, $y = x^3$, respectively, from $x = 0$ to $x = 10$.

174. Work* done by a variable force. — Suppose the force is expressed as a function of some variable, t, and that the displacement is expressed as a function of the same variable. Thus let the force be

$$f = \phi(t)$$

and the displacement in the direction of the force

$$s = F(t).$$

Now the element of work is

$$\Delta w = \Delta s \cdot f + \theta\, \Delta f \cdot \Delta s,$$

where f is ordinate of P, $CP = \Delta f$ and $\theta \gtreqless 1$ (see Fig. 110a), Δf being positive or negative according as the force is an increasing or decreasing function with the displacement. Then

* Work is defined as the product of force by the component of displacement in the direction of the force.

$$\frac{\Delta w}{\Delta t} = \frac{\Delta s}{\Delta t} \cdot f + \theta \frac{\Delta s}{\Delta t} \cdot \Delta f,$$

$$\frac{dw}{dt} = f \frac{ds}{dt}.$$

That is,
$$\frac{dw}{dt} = \phi(t) \cdot F'(t).$$

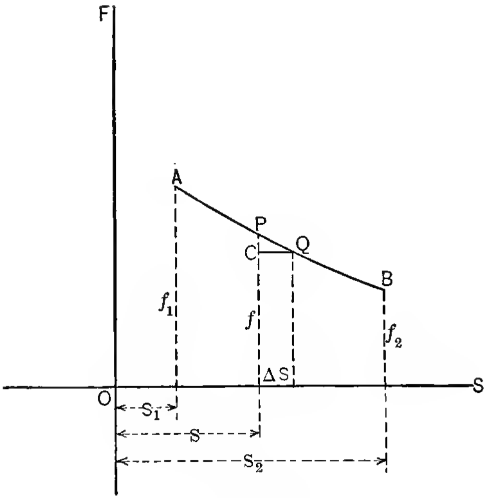


FIG. 110*a*.

Therefore

$$w = \int_{t_1}^{t_2} \phi(t) \cdot F'(t).$$

1. What is the work done by a force $f(x) = 10\,x^2$ in a displacement $\phi(x) = x$, from $x = 0$ to $x = 10$? Write

$$w = \int_0^{10} 10\,x^2 \cdot 1 = \int_0^{10} 10\,x^2 = \frac{10\,x^3}{3}\Bigg]_{x=0}^{x=10} = 3333 \text{ units of work.}$$

2. It requires a force of 5 lbs. to stretch a spring 1″. If the force required for any stretch is proportional to the stretch, find the work done in stretching the spring $1\frac{1}{2}$ ft.

Note. — Write $f(x) = kx$ and $\phi(x) = x$.

3. The force of gravity varies as the square of the distance from the center of the earth. Find the work required to lift a 1-lb. mass from the earth's surface to a point 500 mi. high. It is given that force is 1 lb. at the surface and the radius of the earth 4000 mi.

4. Find the work done by a force $f = \phi(t) = 3t^3 - 8t + 1$ in a displacement $s = F(t) = 8t - 6$, from $t = 0$ to $t = 40$.

5. A force varies inversely as the displacement; when the displacement is 1 the force is 100. Find the work done in a displacement of 100 from a displacement of 5.

6. What work is done in winding, on a windlass, a chain 100′ long, weighing $2\frac{1}{2}$ lbs. to the linear foot?

7. How much work is done by a force that can just roll a barrel weighing 300 lb. up a smooth incline of inclination 30° with the horizontal?

175. So far we have regarded the integral solely as a function whose derivative is given, that is, as the inverse of the derivative. This viewpoint is fundamental and of great value. In order to realize more fully the power and utility of integration as a mathematical instrument we must take another, though not contradictory, viewpoint and look upon an integral as the **sum** of **infinitesimal elements** under certain conditions. This idea is implied in the second form of integrals given in the preceding list of integrals.

Let us now consider the problem of **171.** We have

$$\Delta A = y\,\Delta x + \theta\,\Delta y\,\Delta x. \tag{1}$$

The first term on the right, $y\,\Delta x$, is defined as the **differential** of A or "**differential** A" and written dA. We shall write dx (differential x) for Δx and instead of the above equation we shall consider

$$dA = y\,dx. \tag{2}$$

Note that **differential** A, (dA), is precisely the **derivative** of A **with respect to** x, $\left(\frac{dA}{dx}\right)$, **multiplied by** dx. This principle is general. It is now seen that $y\,dx$ is a rectangle inscribed

under the arc AB and that the area under PQ may be considered the limit of the sum of a set of such inscribed rectangles as their width, dx, approaches zero. This idea is similar to the one used in elementary geometry to show that a pyramid is the limit of the sum of a set of inscribed or circumscribed prisms. See Wentworth, P. and S. Geom. (rev. ed.), p. 312, or some other text.

We may now write

$$(3) \quad A = \int_{x_1}^{x_2} dA = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1),$$

where $f(x)$ is the derivative of $F(x)$. This example shows that integrals considered from the new viewpoint need no rules or processes different from those developed for the integral as the inverse of the derivative. The only difference lies in the construction and interpretation of the integrand.

Example. — Let the student revise, according to the new viewpoint, each of the illustrative problems of **172**, **173**, **174**.

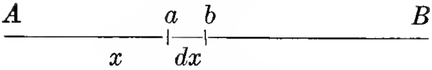
176. We shall now apply the idea of summation to the solution of a new type of problem. In the solution we shall need the idea of moment **81**, Ex. 5.

Note. — The **centroid** of a **system** of **parallel forces** is the point of application of their **resultant** or **equilibrant.** The equilibrant is equal to and opposite in direction to the resultant.

The **center** of **gravity** of a **material body** is the **centroid** of the **forces acting** between its **particles** and the **earth.**

With these definitions we now proceed to find the center of gravity of a thin straight rod AB of unit mass per unit length and length l. Let ab be any small portion of AB of length dx. Its mass and the measure of its attractive force may also be taken as dx, since the density is unity by hypothesis.

Taking A as the origin, AB as the x-axis, l as the length of AB and x as the distance from A to some point of ab, we have for the moment of ab about A,



$$(1) \qquad dM = x\,dx.$$

If we take the sum of the moments of all such elements as $dx \rightarrow 0$, we have

$$(2) \qquad M = \int_0^l dM = \int_0^l x\,dx = \frac{x^2}{2}\bigg]_0^l = \frac{l^2}{2}.$$

The sum of all the forces has the same measure as the mass of the rod. This is, of course, l, which may, for theoretical considerations in later work, be looked upon as

$$(3) \qquad m = \int_0^l dm = \int_0^l dx = x\bigg]_0^l = l.$$

The distance $\bar{x}$ of the point of application of the resultant of the forces from A is equal to the moment M divided by the mass (sum of forces) m or

$$(4) \qquad \bar{x} = \frac{M}{m} = \frac{\int_0^l x\,dx}{\int_0^l dx} = \frac{\frac{l^2}{2}}{l} = \frac{l}{2}.$$

That is, the center of gravity of a thin straight rod is at its center of length, which is what might have been expected from considerations of symmetry. For the value of the above processes in later work the student should thoroughly master them.

Let us now find the centroid of the area under the curve, $y = f(x)$ in Fig. 108. The element of mass (force) is now

$$(5) \qquad dA = y\,dx = f(x)\,dx.$$

The distance of this element from OY (conveniently chosen as the axis of moments) is x. We may now write

$$(6) \qquad dM = x\,dA = xy\,dx = f(x)\,x\,dx$$

as the moment of any element [rectangle under AB]. Hence

$$(7) \qquad M = \int_{x_1}^{x_2} x\,dA = \int_{x_1}^{x^2} xy\,dx = \int_{x_1}^{x_2} f(x)\,x\,dx = \phi(x_2) - \phi(x_1),$$

where $f(x)\,x$ is the derivative of $\phi(x)$. We have therefore

(8) $$\bar{x} = \frac{M}{m} = \frac{\phi(x_2) - \phi(x_1)}{A},$$

where $A = m = F(x_2) - F(x_1)$ and $f(x) = \dfrac{d}{dx} F(x)$. See **171.**

This is the abscissa of the centroid. To find the ordinate we may use the result of the first example and write for the moment of any element $dA = y\,dx$ about OX

(9) $$dM = \frac{y}{2} \cdot dA = \frac{y}{2} \cdot y\,dx = \frac{y^2\,dx}{2} = \frac{[f(x)]^2}{2}\,dx.$$

Whence

(10) $$M = \int_{x_1}^{x_2} dM = \int_{x_1}^{x_2} \frac{y^2}{2}\,dx = \int_{x_1}^{x_2} \frac{(f(x))^2}{2}\,dx$$
$$= \theta(x_2) - \theta(x_1),$$

where $\dfrac{(f(x))^2}{2}$ is the derivative of $\theta(x)$. Therefore

(11) $$\bar{y} = \frac{M}{m} = \frac{\theta(x_2) - \theta(x_1)}{A},$$

where $A = m$ as before.

1. Find the coördinates of the centroid of the area under $y^2 = 12\,x$, from $x = 2$ to $x = 8$.

2. Find the coördinates of the centroid of the area of the right triangle whose vertices are (0, 10), (16, 0), (0, 0).

Note. — This is to be considered as the area under the hypotenuse. Hence find equation of hypotenuse and proceed as above.

3. Find the coördinates of the centroid of the area of the rectangle whose vertices are (1, 0), (6, 0), (1, 7), (6, 7).

Note. — Consider this area as lying under the upper side.

4. Find the distance from the vertex to the centroid of the triangle whose altitude is h and base b.

Note. — Let the origin be the vertex and the base parallel to the y-axis. Take l as the length of any element. Elimi-

nate l in terms of x by use of similar triangles which will naturally occur in the diagram of the problem.

5. Find the distance from the vertex to the centroid of the cone of revolution formed by revolving about OX the right triangle whose vertices are (0, 0), (10, 0), (10, 6).

Note. — The elements are now circular cylinders of altitude dx and radius y, where y is the ordinate of any point on the hypotenuse of the triangle.

6. Find the abscissa of the centroid of the arc of length l and radius r.

Note. — Draw the arc so the origin is the center and the x-axis bisects the arc. The element of arc is da, its abscissa is $r \cos \theta$, where $\theta = a/r$ in radians measured from OX. Take limits from $-\frac{l}{2}$ to $\frac{l}{2}$ or for θ, $-\frac{l}{2r}$ to $\frac{l}{2r}$.

7. Find the centroid of a thin rod whose density varies as its distance from the left end, the density being 5 at unit distance, the rod being 16 units long.

Note. — The element of mass is now $5\,x\,dx$.

8. Solve Ex. 4 above if the density varies as the distance from the vertex and is 3.5 at unit distance.

9. Solve Ex. 5, under same conditions as Ex. 8.

177. Integration by parts. — One of the most useful methods of integration when no formula applies directly is integration by parts. It depends on the possibility of separating the given integrand into two factors one of which can be integrated directly. Consider

(1) $$d\,(uv) = d/dx\,(uv)\,dx = u\,dv - v\,du;$$

whence (2) $$\int u\,dv = uv - \int v\,du,$$

where dv is the integrable factor mentioned above. To apply this formula consider

$$\int x \sin x\,dx.$$

Take $x = u$, $\sin x\,dx = dv$ and substitute in the formula

$$\int x \sin x\,dx = x\,(-\cos x) - \int -\cos x\,dx$$
$$= -x\cos x + \sin x + C.$$

In this example take $\sin x = u$ and $x\,dx = dv$. Then we obtain by substituting in the formula

$$\int x \sin x\,dx = \frac{x^2}{2}\sin x - \int \frac{x^2}{2}\cos x\,dx.$$

The last integrand is more complicated than the original. These results teach us that the choice of factors of the integrand is a matter of vital importance. Sometimes only repeated trials will reveal which set of factors will lead to an integration. Integrate the following:

1. $x \log x\,dx$.
2. $(\sec^2 x - 1)\,x\,dx$.
3. $x \cos x\,dx$.
4. $xe^{ax}\,dx$.
5. $\log x\,dx$.
6. $xe^{x}\,dx$.

178. From **171** it is easily seen that if the arc AB is small the chord AB is nearly equal to it. This idea will enable us to employ integration to determine the length of an arc of a curve when its equation is given. For as in geometry we regard the circle as the limit of the sum of the sides of an inscribed polygon so we now regard the curve PQ as the limit of the sum of such chords as AB. Hence if we write

(1) $$\text{chord } AB = \sqrt{AC^2 + BC^2} = \sqrt{1 + \frac{BC^2}{AC^2}}\,AC,$$

and put $AC = \Delta x$, $BC = \Delta y$ and $AB = \Delta s$ we obtain, noting that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx},$$

(2) $$\text{arc } PQ = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Sometimes it may be desirable to use the form

(3) $$\text{arc } PQ = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy,$$

which is easily seen to be equivalent to the above formula, the difference being that we now integrate with respect to y instead of x and must use y-limits.

1. Find the length of an arc of $y^2 = x^3$ from the origin to the point (4, 8). From the given equation

$$y = x^{\frac{3}{2}},$$
$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}},$$
$$\text{arc} = \int_0^4 ds = \tfrac{1}{2}\int_0 \sqrt{4 + 9x}\,dx = \tfrac{1}{2}\int_0^4 (4 + 9\,x)^{\frac{1}{2}}\,dx$$
$$= \tfrac{1}{27}\,(4 + 9\,x)^{\frac{3}{2}}\Big]_0^4 = 9.34.$$

2. Find the length of the catenary $y = e^{\frac{x}{2}} - e^{-\frac{x}{2}}$ from $x = 0$ to $x = 10$.

3. Find the length of $y = \sin x$ from $x = 0$ to $x = \pi$.

4. Find the length of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a$ from $x = 0$ to $x = a$.

5. Find the length of one arch of the cycloid $\begin{cases} x = a\theta - a\sin\theta. \\ y = a - a\cos\theta. \end{cases}$

Note. — Remember $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$. Use some formulas from **64** to reduce the integrand to simpler form.

179. Very often an integrand may be simplified and made to fit some formula of integration by a transformation of the variable (see Chap. XIII) or what is ordinarily called a substitution. Consider the example $\displaystyle\int \frac{x\,dx}{1 + x^{\frac{1}{2}}}$. Let $x = z^2$. Then $dx = 2\,z\,dz$. Substituting in the given integrand

$$\int \frac{x\,dx}{1 + x^{\frac{1}{2}}} = \int\left(2\,z^2 - 2\,z + 2 - \frac{2}{z + 1}\right)dz$$
$$= \frac{2\,z^3}{3} - z^2 + 2\,z - 2\log(z + 1) + C.$$

Restoring x this becomes

$$\tfrac{2}{3}\,x^{\frac{3}{2}} - x + 2\,x^{\frac{1}{2}} - \tfrac{1}{2}\log(x^{\frac{1}{2}} + 1) + C.$$

Consider another example

$$\int \frac{dx}{\sqrt{1+x^2}}.$$

Let $x = \tan z$. Then $dx = \sec^2 z\, dz$ and $\sqrt{1+x^2} = \sec z$. Making the substitution there results

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \sec z\, dz$$

$$= \int \sec z \frac{(\sec z + \tan z)}{(\sec z + \tan z)} dz = \int \frac{\sec z \tan z + \sec^2 z}{\sec z + \tan z} dz$$

$$= \int \frac{d(\sec z + \tan z)}{\sec z + \tan z} = \log(\sec z + \tan z) + C.$$

Restoring x we obtain

$$\int \frac{dx}{\sqrt{1+x^2}} = \log(x + \sqrt{1+x^2}) + C.$$

Success with the method of substitutions depends upon wide experience. The number of substitution relations is unlimited. The student should consult larger texts on calculus for further treatment.

1. Integrate $\int \frac{dx}{\sqrt{(1+x^2)^3}}$ by the substitution $x = \frac{1}{y}$.

2. Integrate $\int \frac{x^{\frac{1}{4}} - 1}{x^{\frac{1}{2}} + 1} dx$ by the substitution $x = z^4$.

3. Integrate $\int \frac{dx}{\sqrt{1+bx}}$ by the substitution $1 + bx = z^2$.

On p. 276 is a more extensive table of integration formulas than is given in **168**. The purpose of a table of integrals such as this is to save the time of the student and the practicing mathematician after he has sufficient exercise in the process of integration to ensure that he thoroughly understands the significance of the formulas that he is using. The table may be used in solving the following problems. For a more extensive table of integrals the student is referred to Hudson and Lipka's Table of Integrals.

ADDITIONAL DERIVATIVES

1. $$\frac{d}{dx}(\text{arc sin } u) = \frac{\frac{du}{dx}}{\sqrt{1-u^2}} = -\frac{d}{dx}(\text{arc cos } u).$$

2. $$\frac{d}{dx}(\text{arc tan } u) = \frac{\frac{du}{dx}}{1+u^2} = -\frac{d}{dx}(\text{arc tan } u).$$

3. $$\frac{d}{dx}(\text{arc sec } u) = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}} = -\frac{d}{dx}(\text{arc csc } u).$$

4. $$\frac{d}{dx}(\text{arc vers } u) = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}} = -\frac{d}{dx}(\text{arc covers } u).$$

5. From vers $u = 1 - \cos u$ find $\frac{d}{dx}(\text{vers } u) = \sin u \frac{du}{dx}$.

6. From covers $u = 1 - \sin u$ find $\frac{d}{dx}(\text{covers } u) = -\cos u \frac{du}{dx}$.

SUPPLEMENTARY EXERCISES

Find the derivatives of the following:

1. $x^4 + e^{\frac{x}{2}} - \cos x$.

2. $x^3\sqrt{a^2 - x^2} + \frac{\tan x}{x}$.

3. $x^{-\frac{3}{2}} - x \sec x + \frac{1}{\sqrt{1-x^2}}$.

4. $e^{-x^2}(1 - ax)$.

5. $\text{Log}(1 - x^2) - \log \sin x^2$.

6. $3 \sin 3x - a \cos nx$.

7. $(1 - \sin^2 x)^{\frac{3}{2}}$.

8. $\frac{1}{a + b \sin x}$.

9. $e^{-ax} \sin x + \log \frac{1}{1-x^2}$.

10. $\sin x \cos 3x$.

11. $x^4 \sin 5x$.

12. $e^{\sin x}(1 - x^2)$.

13. $\frac{\log x}{x^2}$.

14. $\log(1 - 3x + 4x^2)$.

15. $\log \frac{1-x^2}{3+x}$.

16. $\text{arc sin}(1 - x^2)$.

17. $\text{arc tan } ax^2$.

18. $x^4 \text{ arc cos } x$.

19. $\sin x^{\frac{1}{2}}$.

20. $e^{\text{arc sin } x}$.

21. $\text{arc sec } \frac{1}{x}$.

22. $6 \sin e^{-x^2}$.

Integrate the following:

1. $\int_0^1 x^4\,dx.$

2. $\int_5^{10} \frac{5\,dx}{e^{6x}}.$

3. $\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$

4. $\int_0^{\frac{\pi}{2}} x \sin x\,dx.$

5. $\int_1^2 e^{-\frac{x}{2}}.$

6. $\int (1-x^2)^{\frac{3}{2}} x\,dx.$

7. $\int_0^{\pi} \sin x \cos x\,dx.$

8. $\int \sin^2 x\,dx.$

9. $\int \frac{x\,dx}{e^x}.$

10. $\int x e^{bx}\,dx.$

11. $\int \text{arc sin}\, x\,dx.$

12. $\int x \tan^3 x\,dx.$

13. $\int x^2 \log x\,dx.$

14. $\int x^2\, \text{arc sec}\, x\,dx.$

15. $\int \tan^4 ax\,dx.$

16. $\int \cot^3 x \csc x\,dx.$

17. $\int \frac{\tan^5 x}{\sec^3 x}\,dx.$

18. $\int_0^{\infty} \frac{dx}{1+x^2}.$

19. $\int_0^a \frac{dx}{a^2+x^2}.$

20. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x\,dx.$

21. $\int \sin^2 x \cos^2 x\,dx.$

22. $\int \frac{a(x-a)^3}{cx^2}\,dx$

23. $\int 2\,bx\left(\frac{x^2}{k^2}+1\right)^{\frac{1}{2}} dx.$

24. $\int \frac{\log(1+x)}{1+x}\,dx.$

25. $\int ae^x \cdot x^2\,dx$

26. $\int \frac{5\,dx}{\sqrt{10\,x-25\,x^2}}.$

SUPPLEMENTARY PROBLEMS

1. A train moves out of station A on a straight track. Its distance s in feet from the station is given at any time, until it reached its maximum velocity, by the equation of motion $s = 80\,t^2 + 30\,t$, where t is the time in minutes since the train left the station. Find how far the train travels during the first 3 minutes; the first 2 minutes. Find the average velocity of the train during the period of time:

(a) $t = 2$ until $t = 3$.
(b) $t = 2$ until $t = 2.5$.
(c) $t = 2$ until $t = 2.1$.
(d) $t = 2$ until $t = 2.001$.

Find the exact velocity of the train when $t = 2$ and compare it with the average velocities obtained by the above arithmetical method.

2. The displacement, s, of a body is given by the equation $s = 5 + 2t^2 + t^3$, where s is expressed in feet and t in seconds. Find its average acceleration for 0.001 sec., after the instant $t = 2$ and compare it with the exact value of the acceleration at the beginning of the interval.

3. If $s = \frac{1}{2} gt^2$, where $g = 32.16$ ft./sec² and t is expressed in seconds, find the average velocity for 0.01 after the instant $t = 3$ and compare it with the exact value of the velocity when $t = 3$.

4. If a body moves so that its horizontal and its vertical distances from the starting point are, respectively, $x = 16t^2$ and $y = 4t$, show that the equation of its path is $y^2 = x$ and that its horizontal velocity and vertical velocity are, respectively, $32T$ and 4 at the instant $t = T$.

5. If a ball is constrained to move down a plane inclined at an angle θ with the horizontal, the equation of motion is $s = \frac{1}{2}(g \sin \theta) t^2$. Find that value of θ that will give a maximum velocity for a given value of t.

6. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the dimensions of the strongest beam that can be cut from the log whose diameter is $2a$.

7. In measuring an electric current by means of a tangent galvanometer the percentage error due to a given small error in the reading is proportional to $\tan x + \frac{1}{\tan x}$. Show that this is a minimum when $x = 45°$.

8. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle varies as $\frac{x}{(a^2 + x^2)^{\frac{5}{2}}}$, where x = distance of the magnet from the plane of the circle. Prove that the force is a maximum when $x = a/2$.

9. Find the maximum parallelopiped that can be cut from a sphere if one side of the base is twice the other.

10. Assuming that the current in a voltaic cell is $C = \frac{E}{r + R}$, where E and r are constants representing electromotive force and internal resistance respectively, and R is the external resistance, and that the power given out is $P = RC^2$; show that, if different values are given to R, P will be a maximum when $R = r$.

11. A box is to be made from a square piece of cardboard a inches on a side by cutting out squares from the corners and turning up the sides to form the box. Find the side of the square cut out in order that the volume of the box may be a maximum.

12. A water tank 20 ft. high stands on the top of a scaffolding 125 ft. high. At what distance from the base of the scaffolding should one stand in order that the height of the tank might subtend the largest angle at the eye?

13. If there are n voltaic cells each having an electromotive force of e volts and internal resistance of r ohms, and if x cells are arranged in series and n/x rows in parallel, the current that the battery will send through an external resistance R is given by

$$C = \frac{xe}{\frac{x^2r}{n} + R} \text{ amperes.}$$

If $n = 20$, $e = 1.9$ volts, $r = 0.2$ ohm, $R = 0.25$ ohm; how many cells must be in series to give the greatest possible current?

14. If a body moves so that its horizontal and vertical distances from a point are, respectively, $x = 10\,t$, $y = -16\,t^2 + 10\,t$, find its horizontal speed and its vertical speed. Show that the path is $y = \frac{16\,x^2}{100} + x$ and that the slope of the path is the ratio of the vertical speed to the horizontal speed.

15. A point describing the circle $x^2 + y^2 = 25$ passes through $(3, 4)$ with a velocity of 20 ft. per second. Find its component velocities parallel to the axes.

16. A body moves according to the law $s = \cos(nt + e)$. Show that its acceleration is proportional to the space through which it has moved.

17. Find the expression for acceleration for the motion described by the equation

$$x = e^{at}(c_1 \cos bt + c_2 \sin bt).$$

18. If a body is heated to a temperature θ_1 and then allowed to cool by radiation, its temperature at the time t seconds is given by the equation $\theta = \theta_1 e - at$, where a is a constant. Prove that the rate of cooling is proportional to the temperature.

19. If a point referred to rectangular coördinates moves according to the law $x = a \cos t + b$ and $y = a \sin t + c$, show that its velocity has a constant magnitude.

20. If a point moves according to the law $s = \frac{1}{2}\,gt^2 + v_0t + s_0$, find velocity as a function of t, the acceleration as a function of t, and the velocity as a function of s.

21. For a beam carrying a uniformly distributed load, w, per unit length, and fixed at one end,

$$\frac{d^2y}{dx^2} = \frac{w}{2\,EI}(1 - x)^2.$$

Find y in terms of x.

22. Find expressions for velocity and distance when the acceleration is given by $a = m - nk^2 \cos kt$. Determine the constants of integration, C_1 and C_2, by taking $v = 0$ and $s = 0$ when $t = 0$.

23. In a chemical reaction of the first order, where a is the initial concentration of a substance and x is the amount of substance transformed in a time t, the velocity of the reaction is given by the formula $dx/dt = k(a - x)$. Express k as a function of t and x, and x as a function of k and t.

24. A point has an acceleration expressed by the equation $a = -rw^2 \cos wt$, where r and w are constants. Derive expressions for the velocity and the distance traveled if $s = r$ and $v = 0$ when $t = 0$.

25. If y is the deflection at distance x from the fixed end of uniform beam of length l, fixed at one end and loaded with a weight w at the other, then, if we neglect the weight of the beam,

$$\frac{d^2y}{dx^2} = (1 - x)\frac{w}{EI}.$$

E and I are constants depending upon the material and shape of the beam. It is known that the deflection y and slope dy/dx are 0 at the fixed end where $x = 0$. Find an expression for y in terms of x.

26. If the electromotive force, E.M.F., of an alternating current is represented by a sin curve, any ordinate represents the E.M.F. at that point. Show that $E_a = 0.637\,E$, where E = maximum E.M.F. and E_a = average E.M.F.

27. Another value of importance in the treatment of alternating currents is the square root of the mean square of the ordinates, or the effective E.M.F. Show that $E_e = 0.707\,E$, where E_e = the effective E.M.F.

28. Air expands isothermally (without change of temperature) according to Boyle's law, $pv = c$, where c is a constant. The work done by such an expansion while the volume changes from v_1 to v_2 may be represented by the area under the curve $p = c/v$ and between the ordinates $v = v_1$ and $v = v_2$. Sketch the curve and determine this area. Find the work done if the expansion continues indefinitely, that is, if $v \to \infty$. Give a physical interpretation of your result.

29. The equation representing the adiabatic expansion of a gas is $pv^k = c$, where c is a constant. Find the work done by such an expansion by finding the area under the curve $p = c/v^k$ and between the ordinates $v = v_1$ and $v = v_2$. Find the work done if $v_2 \to \infty$. Give a physical interpretation of your result.

Note. — During the adiabatic expansion of a gas heat is not communicated to nor abstracted from the gas.

30. Derive the four general equations of motion, having given that $s = 0$ and $v = v_0$ when $t = 0$. That is, find v as function of t, s as a function of t, s as a function of v and derive the formula $s = \frac{1}{2}(v_0 + v)\,t$.

TABLE OF INTEGRALS

1. (a) $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C. \qquad n \neq -1.$

   (b) $\int \dfrac{du}{u} = \log u + C.$

   (c) $\int (a+bx)^n\,dx = \dfrac{(a+bx)^{n+1}}{b(n+1)} + C.$

   (d) $\int \dfrac{f'(x)}{f(x)}\,dx = \log f(x) + C.$

   (e) $\int \dfrac{du}{u^n} = \dfrac{u^{1-n}}{1-n} + C.$

2. (a) $\int a^x\,dx = \dfrac{a^x}{\log a} + C.$

   (b) $\int e^x\,dx = e^x + C.$

   (c) $\int e^{nx}\,dx = \dfrac{1}{n}e^{nx} + C.$

3. (a) $\int u\,dv = uv - \int v\,du.$

   (b) $\int xe^x\,dx = e^x(x-1) + C.$

   (c) $\int x^2e^x\,dx = e^x(x^2 - 2x + 2) + C.$

   (d) $\int \log x\,dx = x\log x - x + C.$

4. $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\operatorname{arc\,tan}\dfrac{u}{a} + C \quad \text{or} \quad -\dfrac{1}{a}\operatorname{arc\,ctn}\dfrac{u}{a} + C.$

5. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \operatorname{arc\,sin}\dfrac{u}{a} + C \quad \text{or} \quad -\operatorname{arc\,cos}\dfrac{u}{a} + C.$

6. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\operatorname{arc\,sec}\dfrac{u}{a} + C \quad \text{or} \quad -\dfrac{1}{a}\operatorname{arc\,csc}\dfrac{u}{a} + C.$

7. (a) $\int \dfrac{du}{u^2-a^2} = \dfrac{1}{2a}\log\dfrac{u-a}{u+a} + C.$

   (b) $\int \dfrac{du}{a^2-u^2} = \dfrac{1}{2a}\log\dfrac{a+u}{a-u} + C.$

(c) $\displaystyle\int \frac{du}{(u-a)(u-b)} = \frac{1}{a-b}\log\frac{u-a}{u-b} + C.$

8. (a) $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + C.$

(b) $\displaystyle\int \frac{du}{\sqrt{(u-a)(b-u)}} = 2 \text{ arc sin} \sqrt{\frac{u-a}{b-a}} + C.$

(c) $\displaystyle\int \frac{du}{\sqrt{(u-a)(u-b)}} = 2\log\left(\sqrt{u-a} + \sqrt{u-b}\right) + C.$

9. $\displaystyle\int \sqrt{a^2 - u^2}\, du = \tfrac{1}{2}\left(u\sqrt{a^2-u^2} + a^2 \text{arc sin}\, u/a\right) + C.$

10. $\displaystyle\int \sqrt{u^2 \pm a^2}\, du = \tfrac{1}{2}\left[u\sqrt{u^2 \pm a^2} \pm a^2 \log\left(u + \sqrt{u^2 \pm a^2}\right)\right] + C.$

11. $\displaystyle\int \sin ax\, dx = -\frac{1}{a}\cos ax + C.$

12. $\displaystyle\int \cos ax\, dx = \frac{1}{a}\sin ax + C.$

13. $\displaystyle\int \tan ax\, dx = \frac{1}{a}\log\sec ax + C = -\frac{1}{a}\log\cos ax + C.$

14. $\displaystyle\int \text{ctn}\, ax\, dx = \frac{1}{a}\log\sin ax + C.$

15. $\displaystyle\int \sec ax\, dx = \int \frac{\sec ax \cdot (\tan ax + \sec ax)\, dx}{(\tan ax + \sec ax)}$

$$= \frac{1}{a}\log(\sec ax + \tan ax) + C = \frac{1}{a}\log\tan\left(\frac{x}{4} + \frac{ax}{2}\right) + C;$$

since $\left[(\sec ax + \tan ax) = \dfrac{1 + \sin ax}{\cos ax} = \dfrac{1 - \cos(\pi/2 + ax)}{\sin(\pi/2 + ax)} = \right.$ $\left.\tan \tfrac{1}{2}(\pi/2 + ax)\right]$.

16. $\displaystyle\int \csc ax\, dx = \int \csc ax \cdot \frac{(-\text{ctn}\, ax + \csc ax)}{(-\text{ctn}\, ax + \csc ax)}\, dx$

$$= \frac{1}{a}\log(\csc ax - \text{ctn}\, ax) + C = \frac{1}{a}\log\tan\frac{ax}{2} + C.$$

17. $\displaystyle\int \sin^2 u\, du = \frac{u}{2} - \frac{1}{4}\sin 2u + C.$

18. $\displaystyle\int \cos^2 u\, du = \frac{u}{2} + \frac{1}{4}\sin 2u + C.$

19. $\displaystyle\int \sec^2 u\, du = \tan u + C.$

20. $\displaystyle\int \tan^2 u\, du = \tan u - u + C.$

21. $\displaystyle\int \csc^2 u\, du = -\operatorname{ctn} u + C.$

22. $\displaystyle\int \sec u \cdot \tan u\, du = \sec u + C.$

23. $\displaystyle\int \csc u \cdot \operatorname{ctn} u\, du = \csc u + C.$

24. $\displaystyle\int \frac{du}{\sin^2 u} = \int \csc^2 u\, du.$

25. $\displaystyle\int \frac{du}{\cos^2 u} = \int \sec^2 u\, du.$

26. $\displaystyle\int \frac{du}{\sin u \cos u} = \int \frac{\sec^2 u\, du}{\tan u} = \log \tan u + C.$

27. $\displaystyle\int \sin^3 u\, du = \int (1 - \cos^2 u) \sin u\, du.$

28. $\displaystyle\int \cos^3 u\, du = \int (1 - \sin^2 u) \cos u\, du.$

29. $\displaystyle\int u \sin u\, du = \sin u - u \cos u + C.$

30. $\displaystyle\int u \cos u\, du = u \sin u + \cos u + C.$

31. $\displaystyle\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C.$

32. $\displaystyle\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C.$

33. $\displaystyle\int \operatorname{arc} \sin u\, du = u \operatorname{arc} \sin u + \sqrt{1 - u^2} + C.$

34. $\displaystyle\int \operatorname{arc} \cos u\, du = u \operatorname{arc} \cos u - \sqrt{1 - u^2} + C.$

35. $\displaystyle\int \operatorname{arc} \tan u\, du = u \operatorname{arc} \tan u - \tfrac{1}{2} \log (1 + u^2) + C.$

36. $\displaystyle\int \operatorname{arc} \operatorname{ctn} u\, du = u \operatorname{arc} \operatorname{ctn} u + \tfrac{1}{2} \log (1 + u^2) + C.$

37. $$\int u \operatorname{arc}\sin u\, du = \tfrac{1}{4}\left[(2u^2 - 1)\operatorname{arc}\sin u + u\sqrt{1-u^2}\right] + C.$$

38. $$\int u \operatorname{arc}\cos u\, du = \tfrac{1}{4}\left[(2u^2 - 1)\operatorname{arc}\cos u - u\sqrt{1-u^2}\right] + C.$$

39. $$\int u \operatorname{arc}\tan u\, du = \tfrac{1}{2}\left[(u^2 + 1)\operatorname{arc}\tan u - u\right] + C.$$

40. $$\int u \operatorname{arc}\operatorname{ctn} u\, du = \tfrac{1}{2}\left[(u^2 + 1)\operatorname{arc}\operatorname{ctn} u + u\right] + C.$$

41. $$\int \frac{du}{\sqrt{2au - u^2}} = \operatorname{arc}\operatorname{versin}\frac{u}{a};\ (\operatorname{versin} u = 1 - \cos u).$$

42. $$\int \frac{u\, du}{\sqrt{a^2 \pm u^2}} = \pm\sqrt{a^2 \pm u^2} + C.$$

43. $$\int \frac{u\, du}{\sqrt{u^2 \pm a^2}} = \sqrt{u^2 \pm a^2} + C.$$

44. $$\int \frac{du}{u \cdot \log u} = \int \frac{du/u}{\log u} = \log(\log u) + C.$$

45. $$\int u^n \log u\, du = u^{n+1}\left(\frac{\log u}{n+1} - \frac{1}{(n+1)^2}\right) + C.$$

46. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right).$$

47. $$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\log\frac{a+u}{a-u}.$$

48. $$\int \frac{-du}{u^2 - a^2} = -\frac{1}{2a}\log\frac{u-a}{u+a}.$$

USE OF TABLES

The use of tables requires in general a knowledge of interpolation. The method of interpolation is illustrated in what follows. In making interpolations the corrected result should not contain more significant figures than are given in the table which is used. This often requires the "cutting back" of numbers. For example, in the first illustration below, the product of two differences, $0.055 \times 0.67 = 0.03685$, was cut back to 0.037 in order that the corrected result would not contain more significant figures than that part of the table from which the corrections were made. Note that the last figure retained in the correction was raised one unit. When the part cut off is equal to or greater than one-half the next higher unit, then that unit is increased by one — if less than one-half no change is made in that unit.

TABLE I: **Powers and roots of numbers from 1 to 100.** Assuming that the roots of successive numbers are proportional to their corresponding numbers, we may find the root of a number that does not appear in the table, such as 83.67, by interpolation as follows: From the table

$$\sqrt{84} = 9.165.$$

$$\sqrt{83} = 9.110.$$

A difference of 1 in the numbers causes a difference of 0.055 in their roots. Therefore, a difference of 0.67 in the numbers causes a difference of 0.037 ($= 0.67 \times 0.055$) in the roots.

Therefore, $\sqrt{83.67} = 9.110 + 0.037 = 9.147.$

The root of a decimal fraction such as the cube root of 0.06831 may be obtained from the table as follows:

$$\sqrt[3]{0.06831} = \sqrt[3]{\frac{68.31}{1000}} = \frac{1}{10}\sqrt[3]{68.31}.$$

From the table

$$\sqrt[3]{69} = 4.102.$$

$$\sqrt[3]{68} = 4.082.$$

A difference of 1 in the numbers causes a difference of 0.02 in their roots. Therefore, a difference of 0.31 in the numbers causes a difference of 0.006 in the roots.

Therefore, $\sqrt[3]{68.31} \quad = 4.082 + 0.006 = 4.088$

and $\sqrt[3]{0.06831} = \frac{1}{10}\sqrt[3]{68.31} = 0.4088.$

In this illustration notice that the number was first multiplied by 1000 and the cube root of this new number was found from the table. This root when divided by 10 gave the root sought.

To find the square root of the decimal fraction, 0.06831, multiply the number by 100. Then find the root of this number (6.831) and divide this root (2.613) by 10 and obtain 0.2613 as the square root of 0.06831.

TABLE II: **How to find the logarithm of a number.**

Find log 27.4. The characteristic by rule is 1.

To obtain the mantissa from the table, look for 27 in the column headed N. Opposite 27 and in the column headed by 4 find the number 4378, which is the mantissa with decimal point omitted. Therefore, log 27.4 = 1.4378.

Find log 274.3. The mantissa of this logarithm is not given directly in the table. If we assume that a small change in the number causes a proportional change in the logarithm, then we may proceed by interpolation as follows:

mantissa of log 275 is 0.4393.
mantissa of log 274 is 0.4378.

We observe that a difference of 1 in the number makes a difference of 0.0015 in the logarithm, or a difference of 0.3 in the number makes a difference of $0.0015 \times 0.3 = 0.00045$, or cutting the number back one place, our correction is 0.0005. The logarithm of $274.3 = 2.4378 + 0.0005 = 2.4383$. Time will be saved in interpolating by considering the mantissas for

the moment as whole numbers. Thus, instead of writing 0.0015, write 15, and so forth.

How to find the number corresponding to a given logarithm. — Given log N = 2.4383. To find the number N. Since this problem is the converse of the preceding one, we may trace that problem back. We cannot find the mantissa 0.4383 in the table, but we find 0.4393 and 0.4378 and so forth.

Mechanical interpolations. — In order to facilitate the computation, the tabular difference and the proportional part for the fourth figure of the natural numbers is given at the bottom of the page. The student is advised *not* to use this part of the table until he has learned to interpolate mentally with speed and accuracy. In scientific work one is called upon to use many different tables in which tabular differences and proportional parts are not given. For this reason, the student should learn to be independent of these aids.

In the above problems beginning with the tabular difference $4393 - 4378 = 15$, look at bottom of page under "Tab. Diff." for 15 which is found on the third page of the tables. Opposite 15 and in column headed 3 find 4.5. Add this correction, after cutting it back according to rule, to the mantissa 4378 and obtain 4383, the same mantissa as above with decimal point omitted.

If $\log N = 2.4383$, find the number N, proceeding as follows: Find mantissa in the table nearest less than 4383. It is 4378, which is in column headed 4 and opposite the number 27 in the first column. Hence 0.4378 is the mantissa of the logarithm of 274.

Now
$$4383 - 4378 = 5.$$
$$4393 - 4378 = 15.$$

At bottom of page under "Tab. Diff." opposite 15, find the number nearest 5. It is 4.5 in column headed 3. Therefore, 3 is the next figure (fourth) of the number sought. The decimal point, by rule for characteristic, should be placed so that the integral part of the number will have three digits. Therefore, the number is 274.3.

Use of Tables of Trigonometric Functions

In Table III are given the natural and logarithmic functions of angles for every 10 minutes in the quadrant. The angles less than 45° are found in the left-hand column. The functions are given in same line with the angle. Angles from 45° to 90° are found in the right-hand column. It should be noted that when angles are read on left, function names are to be read at top of page and when angles are read in right column, function names are to be read at bottom of page.

To find the logarithmic sine of 41° 20.′ — Since this angle is less than 45° read the angle in left column. On the line of 41° 20′ in column headed log sine find

$$\log \sin 41° \, 20' = 9.8198.$$

The table is so constructed that the logarithmic functions are 10 larger than their actual values. This will be understood if the logarithm of the natural sine of 41° 20′ is found and its logarithm found in Table II. The adding of 10 to the logarithmic functions is a matter of facility in calculating with them.

To find an angle corresponding to a given function we proceed in a manner similar to finding a number when its logarithm is given.

I. POWERS AND ROOTS OF NUMBERS FROM 1 TO 100

No.	Square.	Cube.	Square root.	Cube root.
1	1	1	1.000	1.000
2	4	8	1.414	1.260
3	9	27	1.732	1.442
4	16	64	2.000	1.587
5	25	125	2.236	1.710
6	36	216	2.450	1.817
7	49	343	2.646	1.913
8	64	512	2.828	2.000
9	81	729	3.000	2.080
10	100	1000	3.162	2.154
11	121	1331	3.317	2.224
12	144	1728	3.464	2.289
13	169	2197	3.606	2.351
14	196	2744	3.742	2.410
15	225	3375	3.873	2.466
16	256	4096	4.000	2.520
17	289	4913	4.123	2.571
18	324	5832	4.243	2.621
19	361	6859	4.359	2.668
20	400	8000	4.472	2.714
21	441	9261	4.583	2.759
22	484	10648	4.690	2.802
23	529	12167	4.796	2.844
24	576	13824	4.899	2.885
25	625	15625	5.000	2.924
26	676	17576	5.099	2.963
27	729	19683	5.196	3.000
28	784	21952	5.292	3.037
29	841	24389	5.385	3.072
30	900	27000	5.477	3.107
31	961	29791	5.568	3.141
32	1024	32768	5.657	3.175
33	1089	35937	5.745	3.208
34	1156	39304	5.831	3.240
35	1225	42875	5.916	3.271
36	1296	46656	6.000	3.302
37	1369	50653	6.083	3.332
38	1444	54872	6.164	3.362
39	1521	59319	6.245	3.391
40	1600	64000	6.325	3.420
41	1681	68921	6.403	3.448
42	1764	74088	6.481	3.476
43	1849	79507	6.557	3.503
44	1936	85184	6.633	3.530
45	2025	91125	6.708	3.557
46	2116	97336	6.782	3.583
47	2209	103823	6.856	3.609
48	2304	110592	6.928	3.634
49	2401	117649	7.000	3.659
50	2500	125000	7.071	3.684

I. POWERS AND ROOTS OF NUMBERS FROM 1 TO 100. (*Cont'd*)

No.	Square.	Cube.	Square root.	Cube root.
51	2601	132651	7.141	3.708
52	2704	140608	7.211	3.733
53	2809	148877	7.280	3.756
54	2916	157464	7.349	3.780
55	3025	166375	7.416	3.803
56	3136	175616	7.483	3.826
57	3249	185193	7.550	3.849
58	3364	195112	7.616	3.871
59	3481	205379	7.681	3.893
60	3600	216000	7.746	3.915
61	3721	226981	7.810	3.937
62	3844	238328	7.874	3.958
63	3969	250047	7.937	3.979
64	4096	262144	8.000	4.000
65	4225	274625	8.062	4.021
66	4356	287496	8.124	4.041
67	4489	300763	8.185	4.062
68	4624	314432	8.246	4.082
69	4761	328509	8.306	4.102
70	4900	343000	8.367	4.121
71	5041	357911	8.426	4.141
72	5184	373248	8.485	4.160
73	5329	389017	8.544	4.179
74	5476	405224	8.602	4.198
75	5625	421875	8.660	4.217
76	5776	438976	8.718	4.236
77	5929	456533	8.775	4.254
78	6084	474552	8.832	4.273
79	6241	493039	8.888	4.291
80	6400	512000	8.944	4.309
81	6561	531441	9.000	4.327
82	6724	551368	9.055	4.345
83	6889	571787	9.110	4.362
84	7056	592704	9.165	4.380
85	7225	614125	9.220	4.397
86	7396	636056	9.274	4.414
87	7569	658503	9.327	4.431
88	7744	681472	9.381	4.448
89	7921	704969	9.434	4.465
90	8100	729000	9.487	4.481
91	8281	753571	9.539	4.498
92	8464	778688	9.592	4.514
93	8649	804357	9.644	4.531
94	8836	830584	9.695	4.547
95	9025	857375	9.747	4.563
96	9216	884736	9.798	4.579
97	9409	912673	9.849	4.595
98	9604	941192	9.900	4.610
99	9801	970299	9.950	4.626
100	10000	1000000	10.000	4.642

II. LOGARITHMS

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010

Tab. diff.	Extra digit.								
	1	2	3	4	5	6	7	8	9
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1
38	3.8	7.6	11.4	15.2	19.0	22.8	26.6	30.4	34.2
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3
36	3.6	7.2	10.8	14.4	18.0	27.6	25.2	28.8	32.4
35	3.5	7.0	10.5	14.0	17.5	21.0	24.5	28.0	31.5
34	3.4	6.8	10.2	13.6	17.0	20.4	23.8	27.2	30.6
33	3.3	6.6	9.9	13.2	16.5	19.8	23.1	26.4	29.7
32	3.2	6.4	9.6	12.8	16.0	19.2	22.4	25.6	28.8
31	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9
30	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0

II. LOGARITHMS (*Continued*)

No.	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445

Tab. diff.	Extra digit.								
	1	2	3	4	5	6	7	8	9
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1
28	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
27	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
26	2.6	5.2	7.8	10.4	13.0	15.6	18.2	20.8	23.4
25	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5
24	2.4	4.8	7.2	9.6	12.0	14.4	16.8	19.2	21.6
23	2.3	4.6	6.9	9.2	11.5	13.8	16.1	18.4	20.7
22	2.2	4.4	6.6	8.8	11.0	13.2	15.4	17.6	19.8
21	2.1	4.2	6.3	8.4	10.5	12.6	14.7	16.8	18.9
20	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1
18	1.8	3.6	5.4	7.2	9.0	10.8	12.6	14.4	16.2
17	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4

II. LOGARITHMS (*Continued*)

No.	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Tab. Diff.	Extra digit.								
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
15	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5
14	1.4	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6
13	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
10	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
8	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
6	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
4	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6

III. NATURAL AND LOGARITHMIC FUNCTIONS

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0°	0.0000	−∞	1.0000	0.0000	0.0000	−∞	∞	∞	90°
10′	.0029	7.4637	1:0000	.0000	.0029	7.4637	343.8	12.5363	50′
20′	.0058	.7648	1.0000	.0000	.0058	.7648	171.9	.2352	40′
30′	.0087	.9408	1.0000	.0000	.0087	.9409	114.6	.0591	30′
40′	.0116	8.0658	0.9999	.0000	.0116	8.0658	85.94	11.9342	20′
50′	.0145	.1627	.9999	.0000	.0145	.1627	68.75	.8373	10′
1°	.0175	.2419	.9998	9.9999	.0175	.2419	57.29	.7581	89°
10′	.0204	.3088	.9998	.9999	.0204	.3089	49.10	.6911	50′
20′	.0233	.3668	.9997	.9999	.0233	.3669	42.96	.6331	40′
30′	.0262	.4179	.9997	.9999	.0262	.4181	38.19	.5819	30′
40′	.0291	.4637	.9996	.9998	.0291	.4638	34.37	.5362	20′
50′	.0320	.5050	.9995	.9998	.0320	.5053	31.24	.4947	10′
2°	.0349	.5428	.9994	.9997	.0349	.5431	28.64	.4569	88°
10′	.0378	.5776	.9993	.9997	.0378	.5779	26.43	.4221	50′
20′	.0407	.6097	.9992	.9996	.0407	.6101	24.54	.3899	40′
30′	.0436	.6397	.9990	.9996	.0437	.6401	22.90	.3599	30′
40′	.0465	.6677	.9989	.9995	.0466	.6682	21.47	.3318	20′
50′	.0494	.6940	.9988	.9995	.0495	.6945	20.21	.3055	10′
3°	.0523	.7188	.9986	.9994	.0524	.7194	19.08	.2806	87°
10′	.0552	.7423	.9985	.9993	.0553	.7429	18.08	.2571	50′
20′	.0581	.7645	.9983	.9993	.0582	.7652	17.17	.2348	40′
30′	.0610	.7857	.9981	.9992	.0612	.7865	16.35	.2135	30′
40′	.0640	.8059	.9980	.9991	.0641	.8067	15.61	.1933	20′
50′	.0669	.8251	.9978	.9990	.0670	.8261	14.92	.1739	10′
4°	.0698	.8436	.9976	.9989	.0699	.8446	14.30	.1554	86°
10′	.0727	.8613	.9974	.9989	.0729	.8624	13.73	.1376	50′
20′	.0756	.8783	.9971	.9988	.0758	.8795	13.20	.1205	40′
30′	.0785	.8946	.9969	.9987	.0787	.8960	12.71	.1040	30′
40′	.0814	.9104	.9967	.9986	.0816	.9118	12.25	.0882	20′
50′	.0843	.9256	.9964	.9985	.0846	.9272	11.83	.0728	10′
5°	.0872	.9403	.9962	.9983	.0875	.9420	11.43	.0580	85°
10′	.0901	.9545	.9959	.9982	.0904	.9563	11.06	.0437	50′
20′	.0929	.9682	.9957	.9981	.0934	.9701	10.71	.0299	40′
30′	.0958	.9816	.9954	.9980	.0963	.9836	10.39	.0164	30′
40′	.0987	.9945	.9951	.9979	.0992	.9966	10.08	.0034	20′
50′	.1016	9.0070	.9948	.9977	.1022	9.0093	9.788	10.9907	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

Read down (left column) — *Read up* (right column)

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
6°	0.1045	9.0192	0.9945	9.9976	0.1051	9.0126	9.514	10.9784	84°
10′	.1074	.0311	.9942	.9975	.1080	.0336	9.255	.9664	50′
20′	.1103	.0426	.9939	.9973	.1110	.0453	9.010	.9547	40′
30′	.1132	.0539	.9936	.9972	.1139	.0567	8.777	.9433	30′
40′	.1161	.0648	.9932	.9971	.1169	.0678	8.556	.9322	20′
50′	.1190	.0755	.9929	.9969	.1198	.0786	8.345	.9214	10′
7°	.1219	.0859	.9925	.9968	.1228	.0891	8.144	.9109	83°
10′	.1248	.0961	.9922	.9966	.1257	.0995	7.953	.9005	50′
20′	.1276	.1060	.9918	.9964	.1287	.1096	7.770	.8904	40′
30′	.1305	.1157	.9914	.9963	.1317	.1194	7.596	.8806	30′
40′	.1334	.1252	.9911	.9961	.1346	.1291	7.429	.8709	20′
50′	.1363	.1345	.9907	.9959	.1376	.1385	7.269	.8615	10′
8°	.1392	.1436	.9903	.9958	.1405	.1478	7.115	.8522	82°
10′	.1421	.1525	.9899	.9956	.1435	.1569	6.968	.8431	50′
20′	.1449	.1612	.9894	.9954	.1465	.1658	6.827	.8342	40′
Read down 30′	.1478	.1697	.9890	.9952	.1495	.1745	6.691	.8255	30′ *Read up*
40′	.1507	.1781	.9886	.9950	.1524	.1831	6.561	.8169	20′
50′	.1536	.1863	.9881	.9948	.1554	.1915	6.435	.8085	10′
9°	.1564	.1943	.9877	.9946	.1584	.1997	6.314	.8003	81°
10′	.1593	.2022	.9872	.9944	.1614	.2078	6.197	.7922	50′
20′	.1622	.2100	.9868	.9942	.1644	.2158	6.084	.7842	40′
30′	.1650	.2176	.9863	.9940	.1673	.2236	5.976	.7764	30′
40′	.1679	.2251	.9858	.9938	.1703	.2313	5.871	.7687	20′
50′	.1708	.2324	.9853	.9936	.1733	.2389	5.769	.7611	10′
10°	.1736	.2397	.9848	.9934	.1763	.2463	5.671	.7537	80°
10′	.1765	.2468	.9843	.9931	.1793	.2536	5.576	.7464	50′
20′	.1794	.2538	.9838	.9929	.1823	.2609	5.485	.7391	40′
30′	.1822	.2606	.9833	.9927	.1853	.2680	5.396	.7320	30′
40′	.1851	.2674	.9827	.9924	.1883	.2750	5.309	.7250	20′
50′	.1880	.2740	.9822	.9922	.1914	.2819	5.226	.7181	10′
11°	.1908	.2806	.9816	.9919	.1944	.2887	5.145	.7113	79°
10′	.1937	.2870	.9811	.9917	.1974	.2953	5.066	.7047	50′
20′	.1965	.2934	.9805	.9914	.2004	.3020	4.989	.6980	40′
30′	.1994	.2997	.9799	.9912	.2035	.3085	4.915	.6915	30′
40′	.2022	.3058	.9793	.9909	.2065	.3149	4.843	.6851	20′
50′	.2051	.3119	.9787	.9907	.2095	.3212	4.773	.6788	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
12°	0.2079	9.3179	0.9781	9.9904	0.2126	9.3275	4.705	10.6725	78°
10′	.2108	.3238	.9775	.9901	.2156	.3336	4.638	.6664	50′
20′	.2136	.3296	.9769	.9899	.2186	.3397	4.574	.6603	40′
30′	.2164	.3353	.9763	.9896	.2217	.3458	4.511	.6542	30′
40′	.2193	.3410	.9757	.9893	.2247	.3517	4.449	.6483	20′
50′	.2221	.3466	.9750	.9890	.2278	.3576	4.390	.6424	10′
13°	.2250	.3521	.9744	.9887	.2309	.3634	4.332	.6366	77°
10′	.2278	.3575	.9737	.9884	.2339	.3691	4.275	.6309	50′
20′	.2306	.3629	.9730	.9881	.2370	.3748	4.219	.6252	40′
30′	.2334	.3682	.9724	.9878	.2401	.3804	4.165	.6196	30′
40′	.2363	.3734	.9717	.9875	.2432	.3859	4.113	.6141	20′
50′	.2391	.3786	.9710	.9872	.2462	.3914	4.061	.6086	10′
14°	.2419	.3837	.9703	.9869	.2493	.3968	4.011	.6032	76°
10′	.2447	.3887	.9696	.9866	.2524	.4021	3.962	.5979	50′
Read down 20′	.2476	.3937	.9689	.9863	.2555	.4074	3.914	.5926	40′ *Read up*
30′	.2504	.3986	.9681	.9859	.2586	.4127	3.867	.5873	30′
40′	.2532	.4035	.9674	.9856	.2617	.4178	3.821	.5822	20′
50′	.2560	.4083	.9667	.9853	.2648	.4230	3.776	.5770	10′
15°	.2588	.4130	.9659	.9849	.2679	.4281	3.732	.5719	75°
10′	.2616	.4177	.9652	.9846	.2711	.4331	3.689	.5669	50′
20′	.2644	.4223	.9644	.9843	.2742	.4381	3.647	.5619	40′
30′	.2672	.4269	.9636	.9839	.2773	.4430	3.606	.5570	30′
40′	.2700	.4314	.9628	.9836	.2805	.4479	3.566	.5521	20′
50′	.2728	.4359	.9621	.9832	.2836	.4527	3.526	.5473	10′
16°	.2756	.4403	.9613	.9828	.2867	.4575	3.487	.5425	74°
10′	.2784	.4447	.9605	.9825	.2899	.4622	3.450	.5378	50′
20′	.2812	.4491	.9596	.9821	.2931	.4669	3.412	.5331	40′
30′	.2840	.4533	.9588	.9817	.2962	.4716	3.376	.5284	30′
40′	.2868	.4576	.9580	.9814	.2994	.4762	3.340	.5238	20′
50′	.2896	.4618	.9572	.9810	.3026	.4808	3.305	.5192	10′
17°	.2924	.4659	.9563	.9806	.3057	.4853	3.271	.5147	73°
10′	.2952	.4700	.9555	.9802	.3089	.4898	3.237	.5102	50′
20′	.2979	.4741	.9546	.9798	.3121	.4943	3.204	.5057	40′
30′	.3007	.4781	.9537	.9794	.3153	.4987	3.172	.5013	30′
40′	.3035	.4821	.9528	.9790	.3185	.5031	3.140	.4969	20′
50′	.3062	.4861	.9520	.9786	.3217	.5075	3.108	.4925	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18°	0.3090	9.4900	0.9511	9.9782	0.3249	9.5118	3.078	10.4882	72°
10′	.3118	.4939	.9502	.9778	.3281	.5161	3.048	.4839	50′
20′	.3145	.4977	.9492	.9774	.3314	.5203	3.018	.4797	40′
30′	.3173	.5015	.9483	.9770	.3346	.5245	2.989	.4755	30′
40′	.3201	.5052	.9474	.9765	.3378	.5287	2.960	.4713	20′
50′	.3228	.5090	.9465	.9761	.3411	.5329	2.932	.4671	10′
19°	.3256	.5126	.9455	.9757	.3443	.5370	2.904	.4630	71°
10′	.3283	.5163	.9446	.9752	.3476	.5411	2.877	.4589	50′
20′	.3311	.5199	.9436	.9748	.3508	.5451	2.850	.4549	40′
30′	.3338	.5235	.9426	.9743	.3541	.5491	2.824	.4509	30′
40′	.3365	.5270	.9417	.9739	.3574	.5531	2.798	.4469	20′
50′	.3393	.5306	.9407	.9734	.3607	.5571	2.773	.4429	10′
20°	.3420	.5341	.9397	.9730	.3640	.5611	2.748	.4389	70°
10′	.3448	.5375	.9387	.9725	.3673	.5650	2.723	.4350	50′
20′	.3475	.5409	.9377	.9721	.3706	.5689	2.699	.4311	40′
30′	.3502	.5443	.9367	.9716	.3739	.5727	2.675	.4273	30′
40′	.3529	.5477	.9356	.9711	.3772	.5766	2.651	.4234	20′
50′	.3557	.5510	.9346	.9706	.3805	.5804	2.628	.4196	10′
21°	.3584	.5543	.9336	.9702	.3839	.5842	2.605	.4158	69°
10′	.3611	.5576	.9325	.9697	.3872	.5879	2.583	.4121	50′
20′	.3638	.5609	.9315	.9692	.3906	.5917	2.561	.4083	40′
30′	.3665	.5641	.9304	.9687	.3939	.5954	2.539	.4046	30′
40′	.3692	.5673	.9293	.9682	.3973	.5991	2.517	.4009	20′
50′	.3719	.5704	.9283	.9677	.4006	.6028	2.496	.3972	10′
22	.3746	.5736	.9272	.9672	.4040	.6064	2.475	.3936	68°
10′	.3773	.5767	.9261	.9667	.4074	.6100	2.455	.3900	50′
20′	.3800	.5798	.9250	.9661	.4108	.6136	2.434	.3864	40′
30′	.3827	.5828	.9239	.9656	.4142	.6172	2.414	.3828	30′
40′	.3854	.5859	.9228	.9651	.4176	.6208	2.395	.3792	20′
50′	.3881	.5889	.9216	.9646	.4210	.6243	2.375	.3757	10′
23°	.3907	.5919	.9205	.9640	.4245	.6279	2.356	.3721	67°
10′	.3934	.5948	.9194	.9635	.4279	.6314	2.337	.3686	50′
20′	.3961	.5978	.9182	.9629	.4314	.6348	2.318	.3652	40′
30′	.3987	.6007	.9171	.9624	.4348	.6383	2.300	.3617	30′
40′	.4014	.6036	.9159	.9618	.4383	.6417	2.282	.3583	20′
50′	.4041	.6065	.9147	.9613	.4417	.6452	2.264	.3548	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log	Angle.
	Consie.		Sine.		Cotangent.		Tangent.		

Read down *Read up*

For angles over 45° use right column and take names of function at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
24°	0.4067	9.6093	0.9135	9.9607	0.4452	9.6486	2.246	10.3514	66°
10′	.4094	.6121	.9124	.9602	.4487	.6520	2.229	.3480	50′
20′	.4120	.6149	.9112	.9596	.4522	.6553	2.211	.3447	40′
30′	.4147	.6177	.9100	.9590	.4557	.6587	2.194	.3413	30′
40′	.4173	.6205	.9088	.9584	.4592	.6620	2.178	.3380	20′
50′	.4200	.6232	.9075	.9579	.4628	.6654	2.161	.3346	10′
25°	.4226	.6259	.9063	.9573	.4663	.6687	2.145	.3313	65°
10′	.4253	.6286	.9051	.9567	.4699	.6720	2.128	.3280	50′
20′	.4279	.6313	.9038	.9561	.4734	.6752	2.112	.3248	40′
30′	.4305	.6340	.9026	.9555	.4770	.6785	2.097	.3215	30′
40′	.4331	.6366	.9013	.9549	.4806	.6817	2.081	.3183	20′
50′	.4358	.6392	.9001	.9543	.4841	.6850	2.066	.3150	10′
26°	.4384	.6418	.8988	.9537	.4877	.6882	2.050	.3118	64°
10′	.4410	.6444	.8975	.9530	.4913	.6914	2.035	.3086	50′
20′	.4436	.6470	.8962	.9524	.4950	.6946	2.020	.3054	40′
30′	.4462	.6495	.8949	.9518	.4986	.6977	2.006	.3023	30′
40′	.4488	.6521	.8936	.9512	.5022	.7009	1.991	.2991	20′
50′	.4514	.6546	.8923	.9505	.5059	.7040	1.977	.2960	10′
27°	.4540	.6570	.8910	.9499	.5095	.7072	1.963	.2928	63°
10′	.4566	.6595	.8897	.9492	.5132	.7103	1.949	.2897	50′
20′	.4592	.6620	.8884	.9486	.5169	.7134	1.935	.2866	40′
30′	.4617	.6644	.8870	.9479	.5206	.7165	1.921	.2835	30′
40′	.4643	.6668	.8857	.9473	.5243	.7196	1.907	.2804	20′
50′	.4669	.6692	.8843	.9466	.5280	.7226	1.894	.2774	10′
28°	.4695	.6716	.8829	.9459	.5317	.7257	1.881	.2743	62°
10′	.4720	.6740	.8816	.9453	.5354	.7287	1.868	.2713	50′
20′	.4746	.6763	.8802	.9446	.5392	.7317	1.855	.2683	40′
30′	.4772	.6787	.8788	.9439	.5430	.7348	1.842	.2652	30′
40′	.4797	.6810	.8774	.9432	.5467	.7378	1.829	.2622	20′
50′	.4823	.6833	.8760	.9425	.5505	.7408	1.817	.2592	10′
29°	.4848	.6856	.8746	.9418	.5543	.7438	1.804	.2562	61°
10′	.4874	.6878	.8732	.9411	.5581	.7467	1.792	.2533	50′
20′	.4899	.6901	.8718	.9404	.5619	.7497	1.780	.2503	40′
30′	.4924	.6923	.8704	.9397	.5658	.7526	1.768	.2474	30′
40′	.4950	.6946	.8689	.9390	.5696	.7556	1.756	.2444	20′
50′	.4975	.6968	.8675	.9383	.5735	.7585	1.744	.2415	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

Read down / *Read up*

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
30°	0.5000	9.6990	0.8660	9.9375	0.5774	9.7614	1.732	10.2386	60°
10′	.5025	.7012	.8646	.9368	.5812	.7644	1.721	.2356	50′
20′	.5050	.7033	.8631	.9361	.5851	.7673	1.709	.2327	40′
30′	.5075	.7055	.8616	.9353	.5890	.7701	1.698	.2299	30′
40′	.5100	.7076	.8601	.9346	.5930	.7730	1.686	.2270	20′
50′	.5125	.7097	.8587	.9338	.5969	.7759	1.675	.2241	10′
31°	.5150	.7118	.8572	.9331	.6009	.7788	1.664	.2212	59°
10′	.5175	.7139	.8557	.9323	.6048	.7816	1.653	.2184	50′
20′	.5200	.7160	.8542	.9315	.6088	.7845	1.643	.2155	40′
30′	.5225	.7181	.8526	.9308	.6128	.7873	1.632	.2127	30′
40′	.5250	.7201	.8511	.9300	6168	.7902	1.621	.2098	20′
50′	.5275	.7222	.8496	.9292	.6208	.7930	1.611	.2070	10′
32	.5299	.7242	.8480	.9284	.6249	.7958	1.600	.2042	58°
10′	.5324	.7262	.8465	.9276	.6289	.7986	1.590	.2014	50′
20′	.5348	.7282	.8450	.9268	.6330	.8014	1.580	.1986	40′
30′	.5373	.7302	.8434	.9260	.6371	.8042	1.570	.1958	30′
40′	.5398	.7322	.8418	.9252	.6412	.8070	1.560	.1930	20′
50′	.5422	.7342	.8403	.9244	.6453	.8097	1.550	.1903	10′
33°	.5446	.7361	.8387	.9236	.6494	.8125	1.540	.1875	57°
10′	.5471	.7380	.8371	.9228	.6536	.8153	1.530	.1847	50′
20′	.5495	.7400	.8355	.9219	.6577	.8180	1.520	.1820	40′
30′	.5519	.7419	.8339	.9211	.6619	.8208	1.511	.1792	30′
40′	.5544	.7438	.8323	.9203	.6661	.8235	1.501	.1765	20′
50′	.5568	.7457	.8307	.9194	.6703	.8263	1.492	.1737	10′
34°	.5592	.7476	.8290	.9186	.6745	.8290	1.483	.1710	56°
10′	.5616	.7494	.8274	.9177	.6787	.8317	1.473	.1683	50′
20′	.5640	.7513	.8258	.9169	.6830	.8344	1.464	.1656	40′
30′	.5664	.7531	.8241	.9160	.6873	.8371	1.455	.1629	30′
40′	.5688	.7550	.8225	.9151	.6916	.8398	1.446	.1602	20′
50′	.5712	.7568	.8208	.9142	.6959	.8425	1.437	.1575	10′
35°	.5736	.7586	.8192	.9134	.7002	.8452	1.428	.1548	55°
10′	.5760	.7604	.8175	.9125	.7046	.8479	1.419	.1521	50′
20′	.5783	.7622	.8158	.9116	.7089	.8506	1.411	.1494	40′
30′	.5807	.7640	.8141	.9107	.7133	.8533	1.402	.1467	30′
40′	.5831	.7657	.8124	.9098	.7177	.8559	1.393	.1441	20′
50′	.5854	.7675	.8107	.9089	.7221	.8586	1.385	.1414	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

Read down — *Read up*

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS (*Continued*)

Angle.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36°	0.5878	9.7692	0.8090	9.9080	0.7265	9.8613	1.376	10.1387	54°
10′	.5901	.7710	.8073	.9070	.7310	.8639	1.368	.1361	50′
20′	.5925	.7727	.8056	.9061	.7355	.8666	1.360	.1334	40′
30′	.5948	.7744	.8039	.9052	.7400	.8692	1.351	.1308	30′
40′	.5972	.7761	.8021	.9042	.7445	.8718	1.343	.1282	20′
50′	.5995	.7778	.8004	.9033	.7490	.8745	1.335	.1255	10′
37°	.6018	.7795	.7986	.9023	.7536	.8771	1.327	.1229	53°
10′	.6041	.7811	.7969	.9014	.7581	.8797	1.319	.1203	50′
20′	.6065	.7828	.7951	.9004	.7627	.8824	1.311	.1176	40′
30′	.6088	.7844	.7934	.8995	.7673	.8850	1.303	.1150	30′
40′	.6111	.7861	.7916	.8985	.7720	.8876	1.295	.1124	20′
50′	.6134	.7877	.7898	.8975	.7766	.8902	1.288	.1098	10′
38°	.6157	.7893	.7880	.8965	.7813	.8928	1.280	.1072	52°
10′	.6180	.7910	.7862	.8955	.7860	.8954	1.272	.1046	50′
20′	.6202	.7926	.7844	.8945	.7907	.8980	1.265	.1020	40′
30′	.6225	.7941	.7826	.8935	.7954	.9006	1.257	.0994	30′
40′	.6248	.7957	.7808	.8925	.8002	.9032	1.250	.0968	20′
50′	.6271	.7973	.7790	.8915	.8050	.9058	1.242	.0942	10′
39°	.6293	.7989	.7771	.8905	.8098	.9084	1.235	.0916	51°
10′	.6316	.8004	.7753	.8895	.8146	.9110	1.228	.0890	50′
20′	.6338	.8020	.7735	.8884	.8195	.9135	1.220	.0865	40′
30′	.6361	.8035	.7716	.8874	.8243	.9161	1.213	.0839	30′
40′	.6383	.8050	.7698	.8864	.8292	.9187	1.206	.0813	20′
50′	.6406	.8066	.7679	.8853	.8342	.9212	1.199	.0788	10′
40°	.6428	.8081	.7660	.8843	.8391	.9238	1.192	.0762	50°
10′	.6450	.8096	.7642	.8832	.8441	.9264	1.185	.0736	50′
20′	.6472	.8111	.7623	.8821	.8491	.9289	1.178	.0711	40′
30′	.6494	.8125	.7604	.8810	.8541	.9315	1.171	.0685	30′
40′	.6517	.8140	.7585	.8800	.8591	.9341	1.164	.0659	20′
50′	.6539	.8155	.7566	.8789	.8642	.9366	1.157	.0634	10′
41°	.6561	.8169	.7547	.8778	.8693	.9392	1.150	.0608	49°
10′	.6583	.8184	.7528	.8767	.8744	.9417	1.144	.0583	50′
20′	.6604	.8198	.7509	.8756	.8796	.9443	1.137	.0557	40′
30′	.6626	.8213	.7490	.8745	.8847	.9468	1.130	.0532	30′
40′	.6648	.8227	.7470	.8733	.8899	.9494	1.124	.0506	20′
50′	.6670	.8241	.7451	.8722	.8952	.9519	1.117	.0481	10′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

Read down (left angle column) — *Read up* (right angle column)

For angles over 45° use right column and take names of functions at bottom of page.

III. NATURAL AND LOGARITHMIC FUNCTIONS *(Continued)*

Angle.	Sine. Nat.	Sine. Log.	Cosine. Nat.	Cosine. Log.	Tangent. Nat.	Tangent. Log.	Cotangent. Nat.	Cotangent. Log.	
42°	0.6691	9.8255	0.7431	9.8711	0.9004	9.9544	1.111	10.0456	48°
10′	.6713	.8269	.7412	.8699	.9057	.9570	1.104	.0430	50′
20′	.6734	.8283	.7392	.8688	.9110	.9595	1.098	.0405	40′
30′	.6756	.8297	.7373	.8676	.9163	.9621	1.091	.0379	30′
40′	.6777	.8311	.7353	.8665	.9217	.9646	1.085	.0354	20′
50′	.6799	.8324	.7333	.8653	.9271	.9671	1.079	.0329	10′
43°	.6820	.8338	.7314	.8641	.9325	.9697	1.072	.0303	47°
10′	.6841	.8351	.7294	.8629	.9380	.9722	1.066	.0278	50′
20′	.6862	.8365	.7274	.8618	.9435	.9747	1.060	.0253	40′
Read down 30′	.6884	.8378	.7254	.8606	.9490	.9772	1.054	.0228	30′ *Read up*
40′	.6905	.8391	.7234	.8594	.9545	.9798	1.048	.0202	20′
50′	.6926	.8405	.7214	.8582	.9601	.9823	1.042	.0177	10′
44°	.6947	.8418	.7193	.8569	.9657	.9848	1.036	.0152	46°
10′	.6967	.8431	.7173	.8557	.9713	.9874	1.030	.0126	50′
20′	.6988	.8444	.7153	.8545	.9770	.9899	1.024	.0101	40′
30′	.7009	.8457	.7133	.8532	.9827	.9924	1.018	.0076	30′
40′	.7030	.8469	.7112	.8520	.9884	.9949	1.012	.0051	20′
50′	.7050	.8482	.7092	.8507	.9942	.9975	1.006	.0025	10′
45°	.7071	.8495	.7071	.8495	1.0000	.0000	1.000	.0000	45°
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle.
	Cosine.		Sine.		Cotangent.		Tangent.		

For angles over 45° use right column and take names of functions at bottom of page.

IV. NAPERIAN LOGARITHMS OF NUMBERS

1 to 9.9

No.	0	1	2	3	4	5	6	7	8	9
1	0.0000	0.0953	0.0182	0.2624	0.3365	0.4055	0.4700	0.5306	0.5878	0.6419
2	0.6931	0.7419	0.7885	0.8329	0.8755	0.9163	0.9555	0.9933	1.0296	1.0647
3	1.0986	1.1314	1.1632	1.1939	1.2238	1.2528	1.2809	1.3083	1.3350	1.3610
4	1.3863	1.4110	1.4351	1.4586	1.4816	1.5041	1.5261	1.5476	1.5686	1.5892
5	1.6094	1.6292	1.6487	1.6677	1.6864	1.7047	1.7228	1.7405	1.7579	1.7750
6	1.7918	1.8083	1.8245	1.8406	1.8563	1.8718	1.8871	1.9021	1.9169	1.9315
7	1.9459	1.9601	1.9741	1.9879	2.0015	2.0149	2.0281	2.0412	2.0541	2.0669
8	2.0794	2.0906	2.1041	2.1163	2.1282	2.1401	2.1518	2.1633	2.1748	2.1861
9	2.1972	2.2083	2.2192	2.2300	2.2407	2.2513	2.2618	2.2721	2.2824	2.2925

10 to 99

No.	0	1	2	3	4	5	6	7	8	9
1	2.3026	2.3979	2.4849	2.5649	2.6391	2.7081	2.7726	2.8332	2.8904	2.9444
2	2.9957	3.0445	3.0910	3.1355	3.1781	3.2189	3.2581	3.2958	3.3322	3.3673
3	3.4012	3.4340	3.4657	3.4965	3.5264	3.5553	3.5835	3.6109	3.6376	3.6635
4	3.6889	3.7136	3.7377	3.7602	3.7842	3.8067	3.8286	3.8501	3.8712	3.8918
5	3.9120	3.9318	3.9512	3.9703	3.9890	4.0073	4.0254	4.0431	4.0604	4.0775
6	4.0943	4.1109	4.1271	4.1431	4.1589	4.1744	4.1897	4.2047	4.2195	4.2341
7	4.2485	4.2627	4.2767	4.2905	4.3041	4.3175	4.3307	4.3488	4.3567	4.3694
8	4.3820	4.3944	4.4067	4.4188	4.4308	4.4427	4.4543	4.4659	4.4773	4.4886
9	4.4998	4.5109	4.5218	4.5326	4.5433	4.5539	4.5643	4.5747	4.5850	4.5951

V. CONVERSION TABLES

Radians to degrees.		Degrees to radians.		Grades to degrees.		Mils to degrees.	
0.1	5° 44′	0° 10′	0.00291	0.1	0° 5.4′	1	0° 3′ 22.5″
0.2	11 28	0 20	0.00582	0.2	0 10.8	2	0 6 45.0
0.3	17 11	0 30	0.00873	0 3	0 16.2	3	0 10 7.5
0.4	22 55	1	0.01745	0.4	0 21.6	4	0 13 30.0
0.5	28 39	2	0.03491	0.5	0 27.0	5	0 16 52.5
0.6	34 23	3	0.05236	0.6	0 32.4	6	0 20 15.0
0.7	40 06	4	0.06981	0.7	0 37.8	7	0 23 37.5
0 8	45 50	5	0.08727	0.8	0 43.2	8	0 27 00.0
0.9	51 34	10	0.17453	0.9	0 48.6	9	0 30 22.5
1.0	57 18	20	0.34907	1.0	0 54.0	10	0 33 45.0
2.0	114 35	30	0.52360	2.0	1 48.0	15	0 50 37.5
3.0	171 53	40	0.69813	3.0	2 42.0	20	1 7 30.0
4.0	229 11	50	0.87266	4.0	3 36.0	25	1 24 22.5
5.0	286 29	57 18	1.00000	5.0	4 30.0	30	1 41 15.0
6.0	343 46	60	1.04720	6.0	5 24.0	35	1 58 7.5
7.0	401 04	90	1.57080	7.0	6 18.0	40	2 15 00.0
8.0	458 22	......	....	8.0	7 12.0	50	2 48 45.0
9.0	515 40	......	....	9.0	8 06.0	60	3 22 30.0
..	....	......	....	10.0	9 00.0	70	3 56 15.0
..	....	......	....	20.0	18 00.0	80	4 30 00.0
..	....	......	....	30.0	27 00.0	90	5 3 45.0
..	....	......	....	40.0	36 00.0	..	....
..	....	......	....	50.0	45 00.0	..	....
..	....	......	....	100.0	90 00.0		....

VI. FUNCTIONS OF ANGLES IN GRADES

Since 100 grades equals a quadrant or 90°, functions of angles greater than 100 grades can be found from this table by use of formulas in Chapter VIII for finding functions of all angles in terms of functions of angles less than 90°.

Angle, grades.	Sine.		Cosine.		Tangent.		Cotangent.		
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	.
0	0.0000	−∞	1.000	10.0000	0.0000	−∞	∞	∞	100
1	.0157	8.1961	0.9997	9.9999	.01571	8.1962	63 66	11.8058	99
2	.0314	.4971	.9995	.9998	.0314	.4973	31.82	.5027	98
3	.0471	.6731	.9989	.9995	.0472	.6736	21.20	.3264	97
4	.0628	.7979	.9979	.9991	.0629	.7988	15.90	.2013	96
5	.0786	.8946	.9970	.9987	.0787	.8960	12.71	.1040	95
6	.0941	.9736	.9955	.9981	.0945	.9756	10.58	.0244	94
7	.1097	9.0403	.9940	.9974	.1104	9.0430	9.057	10.9570	93
8	.1254	.0981	.9922	.9966	1263	.1015	7.916	.8985	92
9	.1409	.1489	.9901	.9957	.1423	.1533	7.026	.8467	91
10	.1564	.1943	.9876	.9946	.1584	.1997	6.314	.8003	90
11	.1719	.2354	.9851	.9935	.1745	.2419	5.729	.7581	89
12	.1874	.2727	.9822	.9922	.1908	.2805	5.242	.7195	88
13	.2028	.3070	.9774	.9909	.2071	.3162	4.828	.6838	87
14	.2181	.3387	.9759	.9894	.2235	.3493	4.474	.6507	86
15	.2335	.3682	.9723	.9878	.2401	.3804	4.166	.6197	85
16	.2487	.3957	.9685	.9861	.2567	.4095	3.895	.5905	84
17	.2639	.4214	.9645	.9843	.2736	.4370	3.655	.5629	83
18	.2790	.4456	.9603	.9824	.2905	.4632	3.442	.5368	82
19	.2940	.4684	.9559	.9804	.3076	.4880	3.251	.5120	81
20	.3091	.4900	.9510	.9782	.3249	.5118	3.077	.4882	80
21	.3239	.5104	.9460	.9759	.3424	.5345	2.921	.4655	79
22	.3388	.5299	.9408	.9735	.3600	.5563	2.778	.4438	78
23	.3535	.5484	.9354	.9710	.3778	.5773	2.647	.4227	77
24	.3681	.5660	.9298	.9684	.3959	.5976	2.526	.4024	76
25	.3827	.5828	.9239	.9656	.4142	.6172	2.414	.3828	75
26	.3972	.5990	9177	.9627	.4327	.6362	2 311	.3638	74
27	.4115	.6144	.9114	.9597	.4515	.6547	2.215	.3453	73
28	.4258	.6292	.9049	.9566	.4705	.6726	2.125	.3274	72
29	.4400	.6434	.8981	.9533	.4899	.6901	2.042	.3100	71
30	.4540	.6571	.8910	.9499	.5096	.7072	1.962	.2928	70
31	.4680	.6702	.8837	.9463	.5294	.7238	1.889	.2762	69
32	.4817	.6828	.8764	.9427	.5498	.7402	1.819	.2598	68
33	.4955	.6950	.8686	9388	5704	.7562	1.753	.2438	67
34	.5091	.7068	.8608	.9349	.5914	.7719	1.690	.2281	66
35	.5225	.7181	.8527	.9308	.6128	.7873	1.632	.2127	65
36	.5358	.7290	.8443	.9265	.6346	.8025	1.576	.1975	64
37	.5490	.7396	.8358	9221	.6569	.8175	1.522	.1825	63
38	.5621	.7498	.8272	.9176	.6797	.8323	1.472	.1678	62
39	.5750	.7597	.8181	.9128	.7028	.8468	1.423	.1532	61
40	.5878	.7692	.8091	.9080	.7266	.8613	1.376	.1387	60
41	.6005	.7785	.7997	.9029	.7508	.8755	1.332	.1245	59
42	.6129	.7874	.7901	.8977	.7757	.8897	1.289	.1103	58
43	.6253	.7961	.7804	.8923	.7950	.9037	1.248	.0963	57
44	.6374	.8044	.7706	.8868	8274	.9177	1.209	.0824	56
45	.6494	.8125	.7605	.8811	8541	.9315	1.171	.0685	55
46	.6613	.8204	.7501	.8751	.8817	.9453	1.134	.0547	54
47	.6730	.8280	.7396	.8690	.9099	.9590	1.099	.0410	53
48	.6845	.8354	.7290	.8627	9391	.9727	1.065	.0273	52
49	.6960	.8426	.7181	.8562	.9692	.9864	1.032	.0137	51
50	.7071	.8495	.7071	.8495	1.0000	10.0000	1.000	10.0000	50
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	Angle, grades.
	Cosine.		Sine.		Cotangent.		Tangent.		

Read down (left column); *Read up* (right column).

For angles over 50 grades read angles on right and take names of columns at bottom of page.

VII. NATURAL SINES AND COSINES — ANGLES EXPRESSED IN MILS

Mils.	Degrees.	Minutes.	Sine.	Cosine.	Mils.
0	0	0	0	1.0000	1600
50	2	48.75	.0490	.9988	1550
100	5	37.50	.0980	.9952	1500
150	8	26.25	.1467	.9892	1450
200	11	15.00	.1951	.9808	1400
250	14	03.75	.2430	.9700	1350
300	16	52.50	.2903	.9569	1300
350	19	41.25	.3369	.9415	1250
400	22	30.00	.3827	.9239	1200
450	25	18.75	.4273	.9040	1150
500	28	07.50	.4714	.8819	1100
550	30	56.25	.5141	.8577	1050
600	33	45.00	.5556	.8315	1000
650	36	33.75	.5957	.8032	950
700	39	22.50	.6344	.7731	900
750	42	11.25	.6716	.7410	850
800	45	00.00	.7071	.7071	800
Mils.	Degrees.	Minutes.	Sine.	Cosine.	Mils.

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(Numbers refer to pages)